

D01. Massive Neutron Stars and the Shapiro Time Delay (50 points)
Part 1 (26 points)

a) (4 points)

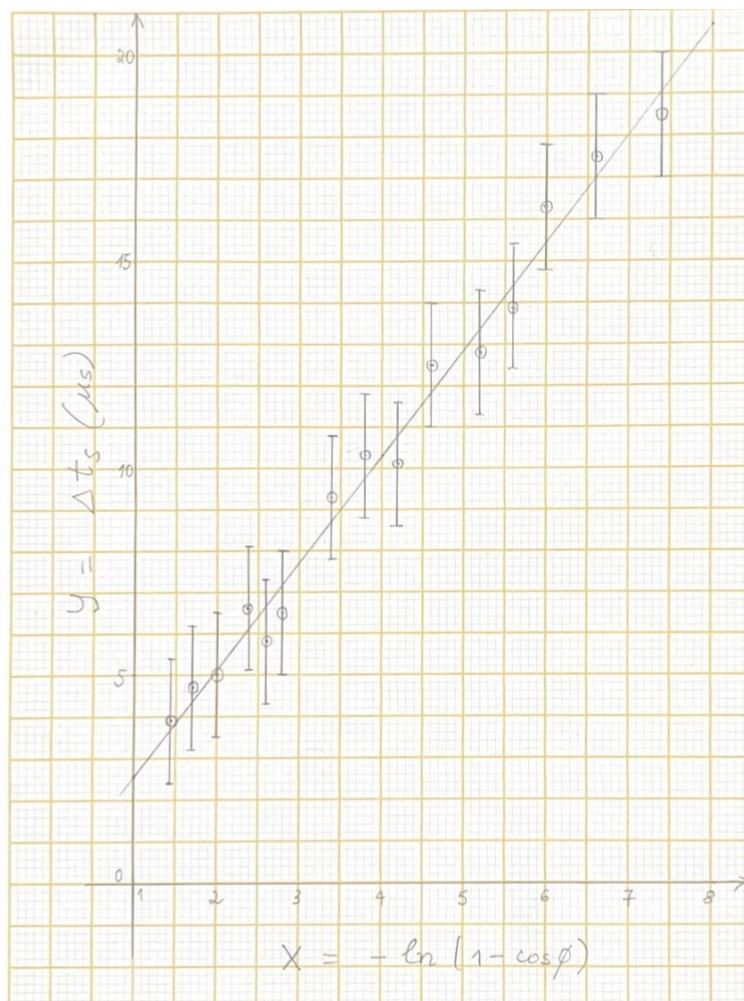
Marking Scheme				Point
$X = -\ln(1 - \cos \phi), \quad Y = \Delta t_S$				1
ID	ϕ (°)	X	Y (μs)	3
1	2.0	7.40	18.5	
2	−3.0	6.59	17.5	
3	4.0	6.02	16.3	
4	−5.0	5.57	13.9	
5	6.0	5.21	12.8	
6	8.0	4.63	12.5	
7	10.0	4.19	10.1	
8	−12.0	3.82	10.3	
9	15.0	3.38	9.3	
10	20.0	2.81	6.5	
11	−22.0	2.62	5.8	
12	25.0	2.37	6.6	
13	30.0	2.01	5.0	
14	−35.0	1.71	4.7	
15	40.0	1.45	3.9	
Each correct row is awarded 0.2 points. Slight deviations due to rounding are acceptable.				

b) (18 points)

Marking Scheme	Point
• Clear axis scaling and labels (X, Y).	1
• Plot correctly the 15 scatter points and their error bars ($\pm 1.5 \mu\text{s}$)	6

- Overlaid a reasonable best-fit line on the plot.

1



Graph D01.1b

$$a = \frac{N \sum XY - \sum X \sum Y}{\Delta} \approx 2.58 \mu\text{s}$$

2

$$b = \frac{\sum X^2 \sum Y - \sum X \sum XY}{\Delta} \approx -0.02 \mu\text{s}$$

2

$$\sigma_a = \sigma_t \sqrt{\frac{N}{\Delta}} = 1.5 \sqrt{\frac{15}{731.95}} \approx 0.21 \mu\text{s}$$

3

$$\sigma_b = \sigma_t \sqrt{\frac{\sum X^2}{\Delta}} = 1.5 \sqrt{\frac{287.04}{731.95}} \approx 0.94 \mu\text{s}$$

3

Note: Students can determine a , b , along with their uncertainties, either by using the formulas for linear regression or from the graph.

Using the graph: $a = [2.4, 2.8]$ and $b = [-0.9, 0.9]$

$$\sigma_a = [0.1, 0.3] \text{ and } \sigma_b = [0.5, 1.5]$$

c) (4 points)

Marking Scheme	Point
$M_{\text{WD}} = \frac{ac^3}{2G}$	1
$M_{\text{WD}} \approx 0.262 M_{\odot}$	1
$\Delta M_{\text{WD}} = M_{\text{WD}} \frac{\sigma_a}{a}$	1
$\Delta M_{\text{WD}} \approx 0.021 M_{\odot}$	1

Part 2 (13 points)

a) (6 points)

Marking Scheme	Point
$P_b^2 = \frac{4\pi^2}{G(M_{\text{PSR}} + M_{\text{WD}})} (a_p + a_{\text{WD}})^3$	2
$a_{\text{WD}} = a_p \frac{M_{\text{PSR}}}{M_{\text{WD}}}$	2
$f_M = \frac{4\pi^2 a_p^3}{GP_b^2}$	1
$f_M \approx 0.002975 M_{\odot}$	1

b) (5 points)

Marking Scheme	Point
$M_{\text{PSR}} = \sqrt{\frac{M_{\text{WD}}^3}{f_M}} - M_{\text{WD}}$	1

$M_{\text{PSR}} \approx 2.20 M_{\odot}$	1
$\frac{\Delta M_{\text{PSR}}}{\Delta M_{\text{WD}}} = \frac{3}{2} \sqrt{\frac{M_{\text{WD}}}{f_M}} - 1 \approx 13.08$	2
$\Delta M_{\text{PSR}} \approx 0.27 M_{\odot}$	1

c) (2 points)

Marking Scheme	Point
$n_{\sigma} = \frac{M_{\text{PSR}} - M_{\text{limit}}}{\Delta M_{\text{PSR}}} = 0.74$	1
No	1

Part 3 (11 points)

a) (3 points)

Marking Scheme	Point
Roemer delay: Achromatic	1
Dispersion measure delay: Chromatic	1
Shapiro delay: Achromatic	1

b) (2 points)

b.1) (1 points)

Marking Scheme	Point
$\Delta t_{\text{DM}} \approx 0.032 \text{ s (32 ms)}$	1

b.2) (1 points)

Marking Scheme	Point
Roemer delay > Dispersion measure delay > Shapiro delay	1

c) (3 points)

Marking Scheme	Point
Roemer delay: B	1
Dispersion measure delay: E	1
Shapiro delay: F	1

d) (3 points)

d.1) (1 point)

Marking Scheme	Point
C	1

d.2) (2 points)

Marking Scheme	Point
B	2

D02. The Nha Trang Expedition (50 points)

Part 1 (15 points)

a) (6 points)

Marking Scheme	Point
$LST = \alpha + H$	1
$\Delta T = LST - T_R$	1
$\Delta T_A = 19^h37^m44.50^s - 19^h36^m53.69^s = 50.81 \text{ s}$	0.5
$\Delta T_B = 21^h47^m42.86^s - 21^h46^m52.11^s = 50.75 \text{ s}$	0.5
$\Delta T_C = 23^h39^m21.93^s - 23^h38^m31.61^s = 50.32 \text{ s}$	0.5
$\Delta T_D = 00^h04^m25.66^s - 00^h03^m35.22^s = 50.44 \text{ s}$	0.5
$\Delta T_{\text{mean}} = \frac{50.81 + 50.75 + 50.32 + 50.44}{4} = 50.58 \text{ s}$	1

$\sigma_T = \left(\frac{0.23^2 + 0.17^2 + 0.26^2 + 0.14^2}{4} \right)^{0.5} = 0.21 \text{ s}$	1
--	---

b) (3 points)

Marking Scheme	Point
$\Delta T_W = \frac{\Delta T_A + \Delta T_B}{2} = \frac{50.81 + 50.75}{2} = 50.78 \text{ s}$	1
$\Delta T_E = \frac{\Delta T_C + \Delta T_D}{2} = \frac{50.32 + 50.44}{2} = 50.38 \text{ s}$	1
$\Delta T_W - \Delta T_E = 50.78 - 50.38 = 0.40 \text{ s}$	1

c) (6 points)

Marking Scheme	Point
$d(\Delta T) = d(LST) = d(\alpha + H) = dH$	1.5
$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H$	1.5
$d(\Delta T) = dH = \frac{\sin z \, dz}{\cos \phi \cos \delta \sin H}$	1
For East stars: $H < 0 \rightarrow \sin H < 0$	0.5
For West stars: $H > 0 \rightarrow \sin H > 0$	0.5
$\frac{1}{2} [d(\Delta T_W) + d(\Delta T_E)] < d(\Delta T_{W,E}) $	1

Part 2 (3 points)

Marking Scheme	Point
$R = \frac{7.888 \times 10^{-5} \times 1011.0}{[(273.0 + 27.0) \times \tan 60^\circ]} = 1.5347 \times 10^{-4} \text{ rad}$	1
$R = 1.5347 \times 10^{-4} \times 206264.8 = 31.66''$	1
$z_{\text{true}} = 30^\circ 00' 00'' + 31.66'' = 30^\circ 00' 31.66''$	1

Part 3 (15 points)

a) (5 points)

Marking Scheme	Point
$\sin \delta = \sin \phi \cos z + \cos \phi \sin z \cos A$	1.5
$\sin z \cos A = \cos \phi \sin \delta - \sin \phi \cos \delta \cos H$	1
$\frac{\sin H}{\sin z} = -\frac{\sin A}{\cos \delta}$	1.5
$\sin A (\cos \phi \, dH) + \cos A \, d\phi = -dz$	1

b) (10 points)

Marking Scheme	Point
Graph D02.3b	

Line A: (0, -2) (10,36)	1.5
Line B: (0,44) (21,0)	1.5
Line C: (0,53) (19,0)	1.5
Line D: (0,14) (-10,0)	1.5
x_0 and y_0 values are within the acceptable ranges: $7'' \leq x_0 \leq 10'', \quad 26'' \leq y_0 \leq 30''$	4

Part 4 (17 points)

a) (4 points)

Marking Scheme	Point
$\phi_{\text{true}} = 12^\circ 15' 00'' + 27.20'' = 12^\circ 15' 27.20''$	1
$dT_{\text{corr}} = \frac{x_0}{15 \cos \phi_0} = \frac{8.50}{15 \times 0.977} = 0.58 \text{ s}$	2
$\Delta T_{\text{true}} = \Delta T_0 + dT_{\text{corr}} = 50.00 + 0.58 = 50.58 \text{ s}$	1

b) (8 points)

Marking Scheme	Point
$d(\Delta T) = \frac{\sin z \, dz}{\cos \phi \cos \delta \sin H}$	1
$\frac{\sin H}{\sin z} = -\frac{\sin A}{\cos \delta}$	1
$d(\Delta T) = -\frac{dz}{\cos \phi \sin A}$	1
$d(\Delta T_E) = -\frac{dz}{\cos \phi}$	1
$d(\Delta T_W) = \frac{dz}{\cos \phi}$	1

$ d(\Delta T_W) - d(\Delta T_E) = \frac{2dz}{\cos \phi}$	1
$\Delta T_W - \Delta T_E = \frac{2dz}{15 \cos \phi}$	1
$dz = \frac{15}{2} (\Delta T_W - \Delta T_E) \cos \phi_0 = 2.93''$	1

c) (2 points)

Marking Scheme	Point
$\varepsilon_A = 8.50'' \sin(285.0^\circ) + 27.20'' \cos(285.0^\circ) + 0.40'' = 0.77''$	0.5
$\varepsilon_B = 8.50'' \sin(245.0^\circ) + 27.20'' \cos(245.0^\circ) + 18.77'' = 0.43''$	0.5
$\varepsilon_C = 8.50'' \sin(70.0^\circ) + 27.20'' \cos(70.0^\circ) - 18.08'' = 0.79''$	0.5
$\varepsilon_D = 8.50'' \sin(125.0^\circ) + 27.20'' \cos(125.0^\circ) + 8.15'' = 0.49''$	0.5

d) (3 points)

Marking Scheme	Point
$d = \sqrt{x_0^2 + y_0^2} = \sqrt{8.50''^2 + 27.20''^2} = 28.50''$	2
$d = 28.50'' \times 31 \text{ m}/1'' \approx 883 \text{ m}$	1

D03. The Mystery of "Little Red Dots" (50 points)

Part 1 (28 points)

a) (5 points)

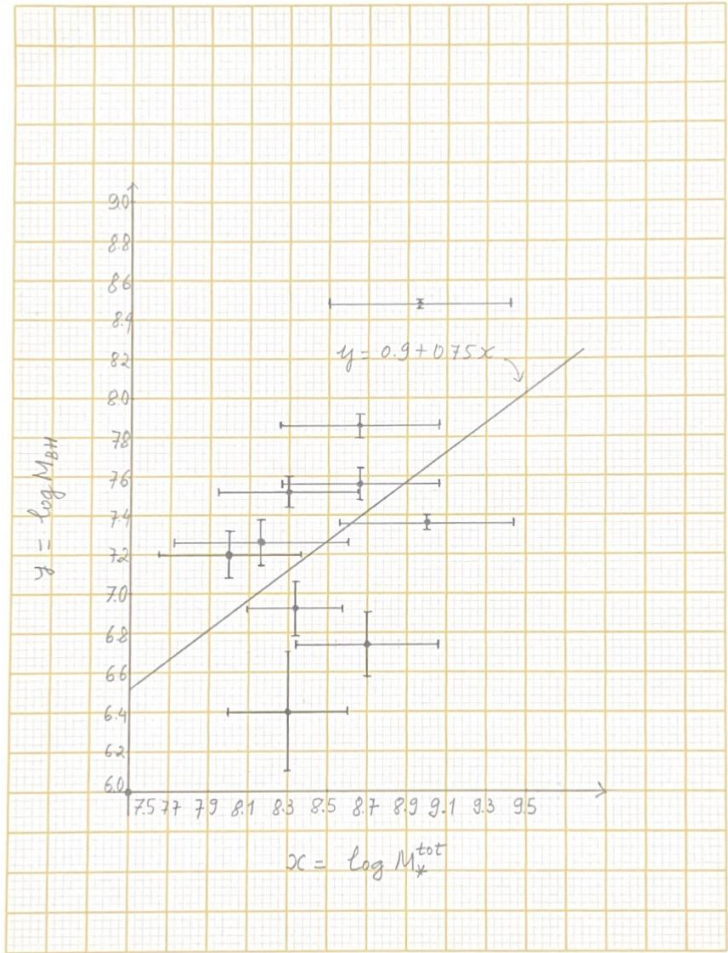
Marking Scheme				Point
LRD ID	$\Delta \log M_*^{\text{tot}} (\Delta x_i)$	$\Delta \log M_{\text{BH}} (\Delta y_i)$	Weight (w_i)	
1	0.24	0.14	12.95	
2	0.30	0.30	5.56	
3	0.37	0.12	6.61	

LRD ID	$\Delta \log M_*^{\text{tot}} (\Delta x_i)$	$\Delta \log M_{\text{BH}} (\Delta y_i)$	Weight (w_i)
4	0.46	0.02	4.72
5	0.36	0.17	6.31
6	0.43	0.11	5.08
7	0.43	0.03	5.38
8	0.39	0.08	6.31
9	0.35	0.08	7.76
10	0.41	0.06	5.82

Note: 0.5 points for each corrected row.

5

b) (12 points)

Marking Scheme	Point
 Graph D03.1b	

Clear axis scaling and labels (x, y).	2
Plotting of the 10 scatter points (0.5 points per each).	5
Correct inclusion of error bars (both axes) for all points (0.5 points per each).	5

c) (7 points)

Marking Scheme	Point
$S_w = \sum w_i \approx 66.49$	0.5
$S_x = \sum w_i x_i \approx 563.16$	0.5
$S_y = \sum w_i y_i \approx 483.65$	0.5
$S_{xx} = \sum w_i x_i^2 \approx 4775.63$	0.5
$S_{xy} = \sum w_i x_i y_i \approx 4100.80$	0.5
$\Delta = S_w S_{xx} - (S_x)^2 \approx 405.96$	0.5
<i>Note: Accept deviations in the final sums due to students truncating w_i floating points. Accept deviations up to ± 5.0 for large sums like S_{xx} and S_{xy}, and up to ± 0.5 for smaller sums like S_w and S_x.</i>	
$\alpha = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta} \approx 0.9$	0.5
$\beta = \frac{S_w S_{xy} - S_x S_y}{\Delta} \approx 0.75$	0.5
$\Delta\alpha = \sqrt{\frac{S_{xx}}{\Delta}} \approx 3.4$	0.5
$\Delta\beta = \sqrt{\frac{S_w}{\Delta}} \approx 0.40$	0.5

$\alpha = 0.9 \pm 3.4, \quad \beta = 0.75 \pm 0.40$ <i>Note: If students report $\alpha = 1 \pm 3$ and $\beta = 0.8 \pm 0.4$, the marks remain unchanged.</i> Best-fit Line Plotting: Draw the line $y = 0.9 + 0.75x$ through the scatter plot.	2
---	---

d) (4 points)

d.1) (2 points)

Marking Scheme	Point
A	2

d.2) (2 points)

Marking Scheme	Point
D	2

Part 2 (6 points)

a) (2 points)

Marking Scheme	Point
$M_*^{\text{tot}} \left(\frac{r_h}{r_h + a} \right) = M_{\text{BH}}$	1
$r_h = a \frac{\frac{M_{\text{BH}}}{M_*^{\text{tot}}}}{1 - \frac{M_{\text{BH}}}{M_*^{\text{tot}}}}$	1

b) (4 points)

Marking Scheme	Point
$\frac{M_{\text{BH}}}{M_*^{\text{tot}}} \approx 0.32$	1
$\frac{r_h}{a} \approx 0.48$	1

C	2
---	---

Part 3 (16 points)

a) (7 points)

Marking Scheme	Point
$F_{\text{center}} = \frac{GmM_{\text{BH}}}{r^2}$	0.5
$F_{\text{near}} = \frac{GmM_{\text{BH}}}{(r - R_*)^2}$	0.5
$\Delta F = \frac{GmM_{\text{BH}}}{(r - R_*)^2} - \frac{GmM_{\text{BH}}}{r^2}$	1
$\Delta F \approx \frac{2GmM_{\text{BH}}R_*}{r^3}$	1
$\frac{2GmM_{\text{BH}}R_*}{r_t^3} = \frac{Gmm_*}{R_*^2}$	1
$r_t = R_* \left(\frac{2M_{\text{BH}}}{m_*} \right)^{\frac{1}{3}}$	1
$M_{\text{H}} = \frac{1}{2} \left(\frac{c^2 R_*}{Gm_*^{1/3}} \right)^{3/2}$	1
$M_{\text{H}} \approx 1.62 \times 10^8 M_{\odot}$	1

b) (4 points)

Marking Scheme	Point
$\log M_{*,\text{max}}^{\text{tot}} = \frac{\log M_{\text{H}} - \alpha}{\beta}$	1
Upper bound: $\log M_{*,\text{max}}^{\text{tot}} \approx 31$	1
Lower bound: $\log M_{*,\text{max}}^{\text{tot}} \approx 3.4$	1
B	1

c) (5 points)

Marking Scheme	Point
$\text{Rate}_{\text{obs}} = \frac{N \times \langle \dot{N}_{\text{TDE}} \rangle}{1 + z}$	2
$\text{Rate}_{\text{obs}} \approx 34 \text{ events/year}$	1.5
$\text{Area} = 15\% \text{ of the celestial sphere}$	1.5