

INSTRUCTIONS

- You are given an envelope marked **B**. Please verify the Contestant Code printed on it. Inside you will find:
 - This Instruction Sheet in English and/or your native language (if applicable)
 - Data Sheet in English and/or your native language (if applicable)
 - A set of Graph Papers
 - A set of Working Sheets
 - One envelope marked **S** (please verify the Contestant Code marked on it).
- You are allowed to open the envelope **S** only when the whistle is blown to signal the start of examination.**
- Inside the envelope **S** you will find the following:
 - Question Paper in English and/or your native language (if applicable)
 - A set of Summary Answer Sheets.
- The total duration of this examination is 4 hours.
- This examination consists of the following questions:

D01	50 points
D02	50 points
D03	50 points
Total:	150 points

- Instructions for writing your answers:**
 - Summary Answer Sheet:

- A separate Summary Answer Sheet is provided for each question. Verify your Contestant Code on each page of the Summary Answer Sheet.
- The final answer(s) to each question or each part of a question must be written in the corresponding box in the Summary Answer Sheet.

2. Graph Paper(s):

- Please write your Contestant Code on each Graph Paper.
- For each Graph Paper that you use, write the graph number (indicated in the question) in the space provided. **Failure to do so may result in the graph not being evaluated.**
- You may ask for one extra Graph Paper, if required.

3. Working Sheets:

- Your detailed work must be shown on the Working Sheet.
- Write your Contestant Code on the front page of each Working Sheet.
- Start a new question on a fresh Working Sheet. Write the question number in the box provided at the top.
- Write only on the front page of the Working Sheet (inside the boxed area). Do not write on the back page.
- For rough work that you do not want to be evaluated, clearly strike that part out.
- For each question, write the sheet number in the box provided at the top front page of the Working Sheet, starting with 1. For example, if you use 1 Working Sheet for D01, write "D01, 1" on that sheet. If you use 2 Working Sheets for D02, write "D02, 1" on the first sheet, and "D02, 2" on the second sheet. The total sheets used for each question must be filled.
- You may ask for extra Working Sheets, if required.
- **Note that failure to mark the Working Sheets, as instructed above, may result in your answers not being evaluated.**

• **At the end of the examination:**

- Ensure your Contestant Code and graph/sheet numbers are correctly written on all used Graph Papers and Working Sheets.
- Arrange your sheets in the following way:
 - Arrange the Summary Answer Sheets in numerical order from D01 to D03. All Summary Answer Sheets must be submitted, whether used or unused.
 - Arrange the Graph Papers in sequential order of the question numbers. Only include the Graph Papers that you actually used.
 - Arrange the Working Sheets in sequential order of the question numbers. Only include the Working Sheets that you actually used.
 - The bundle of Summary Answer Sheets must be placed on top, followed by the bundle of Graph Papers, and then the bundle of Working Sheets.

- Place all the sheets arranged in the above manner inside the envelope **S**.
- **Only the contents of the envelope **S** will be evaluated. Any sheet that is not placed inside this envelope will not be evaluated.**
- Place the envelope **S** inside the envelope **B**.
- Place all other sheets, including the Instruction Sheet(s), Data Sheet(s), Question Paper(s), unused Graph Paper(s), and unused Working Sheet(s) directly inside the envelope **B**.
- Tie the thread seal on the envelope **B**.
- Hand over the envelope **B** to the examiner. Make sure that no sheet is left on the table, except the green and the red cards.
- **General instructions:**
 - If you need to use the toilet during the examination, please raise your hand while holding the green card provided on your table. You will be escorted by a volunteer.
 - If you need any other help during the examination, please raise your hand while holding the red card provided on your table.
 - Use only black or blue pen for writing. For drawing, you may use a pencil.
 - For drawing graphs, you must use a dark pencil.
 - Some points may be deducted if the final answer is given without a detailed solution.
 - Some points may be deducted if the final answers have an inappropriate number of significant digits, no units or incorrect units.
 - The necessary values of universal constants and astronomical data should be taken from the Data Sheet provided to you.

UNIVERSAL CONSTANTS

Avogadro constant	$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$
Electron charge magnitude	$e = 1.602 \times 10^{-19} \text{ C}$
Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m s}^{-1}$
Universal gravitational constant	$G = 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2$
Mass of electron	$m_e = 9.109 \times 10^{-31} \text{ kg} = 0.511 \text{ MeV}/c^2$
Mass of proton	$m_p = 1.673 \times 10^{-27} \text{ kg} = 938.272 \text{ MeV}/c^2$
Mass of He nucleus	$m_{\text{He}} = 6.645 \times 10^{-27} \text{ kg} = 3727.181 \text{ MeV}/c^2$

ASTRONOMICAL DATA

Mass of Sun	$M_{\odot} = 1.988 \times 10^{30} \text{ kg}$
Radius of Sun	$R_{\odot} = 6.957 \times 10^8 \text{ m}$
Luminosity of Sun	$L_{\odot} = 3.828 \times 10^{26} \text{ W}$
Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m s}^{-1}$
Distance from Earth to Sun	$d_{\odot} = 1.496 \times 10^{11} \text{ m}$
Solar constant on Earth	$F_{\odot} = 1361 \text{ W m}^{-2}$
Hubble constant	$H_0 = 67.7 \text{ km s}^{-1} \text{ Mpc}^{-1}$
Age of the Universe today	$t_0 = 13.8 \text{ Gyr}$
Astronomical Unit	$1 \text{ au} = 1.496 \times 10^{11} \text{ m}$
parsec	$1 \text{ pc} = 3.086 \times 10^{16} \text{ m}$
light-year	$1 \text{ ly} = 9.461 \times 10^{15} \text{ m}$

MATHEMATICAL FORMULAS

Derivatives	$y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$
Integrals	$\int y dx = f$ so that $\frac{df}{dx} = y$ and $\int_a^b y dx = f(b) - f(a)$
Chain rule	$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$
Product rule	$\frac{d}{dx} (uv) = \frac{du}{dx} v + u \frac{dv}{dx}$
Quotient rule	$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{1}{v^2} \left(\frac{du}{dx} v - u \frac{dv}{dx} \right)$
Powers	$\frac{d}{dx} x^\alpha = \alpha x^{\alpha-1}$ and $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + \text{constant}$ ($\alpha \neq -1$)
Exponents	$\frac{d}{dx} e^x = e^x$ and $\int e^x dx = e^x + \text{constant}$
Logarithms	$\frac{d}{dx} \ln x = \frac{1}{x}$ and $\int \frac{1}{x} dx = \ln x + \text{constant}$
Sines	$\frac{d}{dx} (\sin x) = \cos x$, $\int \sin x dx = -\cos x + \text{constant}$
Cosines	$\frac{d}{dx} (\cos x) = -\sin x$, $\int \cos x dx = \sin x + \text{constant}$
Taylor expansion up to second orders	$y = y_0 + y'_0 (x - x_0) + \frac{1}{2} y''_0 (x - x_0)^2$ for $x \approx x_0$

D01. Massive Neutron Stars and the Shapiro Time Delay (50 points)

The equation of state (EoS) of ultra-dense matter in neutron stars remains a major unsolved problem in fundamental physics. Theoretical models predicting a “soft” EoS dictate that neutron stars must collapse into black holes before reaching a mass of $2.0 M_{\odot}$. Consequently, a robust mass measurement of a neutron star exceeding $2.0 M_{\odot}$ would definitively rule out soft EoS models.

Radio observations of the millisecond pulsar PSR J0740+6620, located in a binary system with a white dwarf companion, offer a unique testbed for this prediction. According to General Relativity, as the pulsar passes through superior conjunction, when it is directly behind its companion, the companion's gravitational field curves spacetime, delaying the propagation of radio pulses to Earth. This phenomenon is known as the Shapiro time delay. Assuming an edge-on orbital inclination ($i \approx 90^\circ$) and a perfectly circular orbit (eccentricity $e = 0$), the Shapiro delay Δt_S is approximated by

$$\Delta t_S \approx \left(\frac{2GM_{\text{WD}}}{c^3} \right) [-\ln(1 - \cos \phi)] + b$$

where M_{WD} is the white dwarf mass, ϕ is the orbital phase ($\phi = 0^\circ$ at superior conjunction), and b is a baseline timing offset.

The binary system has an orbital period of $P_b = 4.767$ days. The pulsar has a projected semi-major axis of $x = 3.978$ light-seconds, which corresponds to an orbital radius $a_p = cx$. Assume the uncertainties on P_b and x are negligible.

The following table presents 15 observations of pulsar Shapiro delay at various orbital phases (ϕ) near superior conjunction. A constant measurement uncertainty of $\sigma_t = \pm 1.5 \mu\text{s}$ is assumed for all observations.

ID	Orbital Phase ϕ ($^\circ$)	Shapiro Delay Δt_S (μs)
1	2.0	18.5
2	-3.0	17.5
3	4.0	16.3
4	-5.0	13.9
5	6.0	12.8
6	8.0	12.5
7	10.0	10.1
8	-12.0	10.3

ID	Orbital Phase ϕ ($^{\circ}$)	Shapiro Delay Δt_s (μs)
9	15.0	9.3
10	20.0	6.5
11	-22.0	5.8
12	25.0	6.6
13	30.0	5.0
14	-35.0	4.7
15	40.0	3.9

Part 1 (26 points): Estimation of White Dwarf Mass

- a) (4 points) Define appropriate linearized variables (X and Y), and tabulate their values for 15 data points.
- b) (18 points) Plot a scatter diagram of the 15 data points including their available error bars. Overlay the best-fit line on your graph (mark your graph as "D01.1b"). Determine the slope a , the intercept b , and their respective uncertainties (σ_a, σ_b).
- c) (4 points) Determine the mass of the white dwarf (M_{WD}) and its absolute uncertainty (ΔM_{WD}) in units of $M_{\odot}$.

Part 2 (13 points): Binary Dynamics and Error Propagation

- a) (6 points) The mass function of the binary system is defined as

$$f_M = \frac{M_{WD}^3}{(M_{PSR} + M_{WD})^2}$$

where M_{PSR} is the mass of the pulsar. Derive the expression of f_M in terms of the observables a_p and P_b , and hence compute its value in units of $M_{\odot}$.

- b) (5 points) Determine the mass of the pulsar (M_{PSR}) and derive its uncertainty (ΔM_{PSR}).
- c) (2 points) Determine how many standard deviations (σ) the measured pulsar mass lies above the upper mass limit predicted by the "soft EoS" model. Hence, state whether the model can be ruled out.

Part 3 (11 points): Pulse Time of Arrival (TOA): Effect of Multiple Factors

In binary pulsar timing, an observed Time of Arrival (TOA) is inherently shifted by various physical effects relative to an idealized baseline. To accurately determine the companion mass, astrophysicists must account for three primary components of the timing delay:

- Shapiro delay (Δt_S), caused by spacetime curvature around the white dwarf companion, produces a minuscule shift of $\sim 10 \mu\text{s}$.
- Roemer delay (Δt_R): The variation in pulse arrival time caused by the movement of the pulsar along its orbit around the binary's center of mass.
- Dispersion measure delay (Δt_{DM}): The propagation delay experienced by a signal traveling through the ionized interstellar medium. This delay is given by

$$\Delta t_{DM} = K \cdot DM \cdot \frac{1}{f^2}$$

where f is the frequency of the observed signal, K is the dispersion constant, and DM is the dispersion measure.

a) (3 points) Physical classification: Classify each of the following signal delay mechanisms as achromatic (frequency-independent) or chromatic (frequency-dependent).

Roemer delay: ☐ Achromatic ☐ Chromatic

Dispersion measure delay: ☐ Achromatic ☐ Chromatic

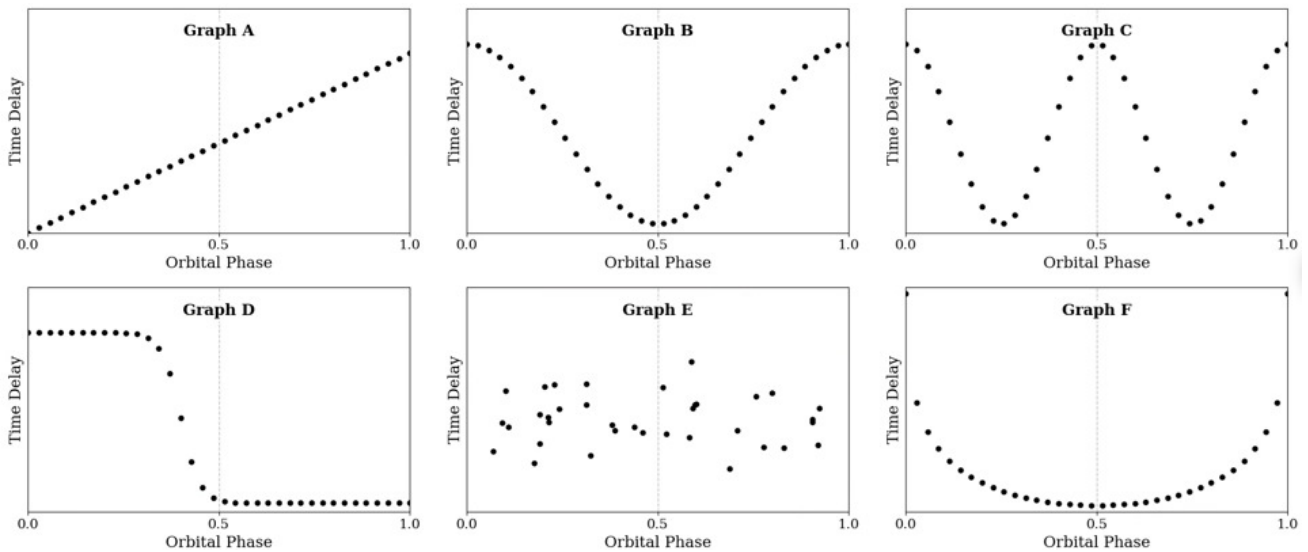
Shapiro delay: ☐ Achromatic ☐ Chromatic

b) (2 points) To understand the difficulty of isolating the Shapiro delay, we must compare the magnitudes of these effects. Given an observing frequency $f = 1400 \text{ MHz}$ (standard L-band), an average interstellar dispersion measure $DM = 15 \text{ pc cm}^{-3}$, and the dispersion constant $K = 4.15 \times 10^3 \text{ s MHz}^2 \text{ pc}^{-1} \text{ cm}^3$:

b.1) (1 points) Calculate Δt_{DM} .

b.2) (1 points) Arrange the three delays (Δt_R , Δt_{DM} , Δt_S) in order from largest to smallest.

c) (3 points) Below are six schematic graphs (A–F) illustrating the time delay as a function of orbital phase over one complete orbit. The vertical axis represents the delay amplitude (numerical values omitted). Match each of the three physical delay mechanisms listed below to the graph that best represents its expected shape.



Roemer delay [?]

Dispersion measure delay [?]

Shapiro delay [?]

d) (3 points) Observational strategy: Variations in the interstellar medium cause the dispersion measure to fluctuate over time, introducing timing noise. Select one correct answer to each question below.

d.1) (1 point) Why is a monochromatic observation (observing at a single frequency) insufficient to constrain the extremely small $10 \mu s$ Shapiro delay?

- A) The telescope's receiver cannot process data faster than $10 \mu s$.
- B) Single frequencies are completely absorbed by the interstellar plasma.
- C) The system contains multiple sources of time delay, so the different physical effects cannot be distinguished from a single observation.
- D) Monochromatic signals do not experience gravitational time dilation.

d.2) (2 points) Which observational strategy is used to measure and correct for dispersion measure variations?

- A) Integrating the signal over a one-year period to average out the dispersion measure variations.
- B) Conducting simultaneous observations at two distinct radio frequencies.

C) Observing strictly at an extremely low frequency where the dispersion measure can be reduced.

D) Only observing the pulsar when it is outside superior conjunction.

D02. The Nha Trang Expedition (50 points)

In 1910, the scientist Alexandre Yersin (discoverer of the bubonic plague bacillus) established an astronomical observatory on the roof of his house in Nha Trang, Vietnam. There, he used a Claude-Driencourt prism astrolabe, designed to record the instant at which a star reaches a fixed zenith distance ($z = 30.0^\circ$). The following data table is inspired by Yersin's observations under atmospheric conditions of $T = 27.0^\circ\text{C}$ and $P = 1011.0\text{ hPa}$.

Star	Right Ascension (α)	Hour Angle (H)	Recorded Time (T_R)	Azimuth (A)
A	$17^{\text{h}}35^{\text{m}}35.55^{\text{s}}$	$+2^{\text{h}}02^{\text{m}}08.95^{\text{s}}$	$19^{\text{h}}36^{\text{m}}53.69^{\text{s}}$	285.0°
B	$19^{\text{h}}59^{\text{m}}52.48^{\text{s}}$	$+1^{\text{h}}47^{\text{m}}50.38^{\text{s}}$	$21^{\text{h}}46^{\text{m}}52.11^{\text{s}}$	245.0°
C	$01^{\text{h}}39^{\text{m}}52.31^{\text{s}}$	$-2^{\text{h}}00^{\text{m}}30.38^{\text{s}}$	$23^{\text{h}}38^{\text{m}}31.61^{\text{s}}$	70.0°
D	$01^{\text{h}}41^{\text{m}}38.84^{\text{s}}$	$-1^{\text{h}}37^{\text{m}}13.18^{\text{s}}$	$00^{\text{h}}03^{\text{m}}35.22^{\text{s}}$	125.0°

Note: Note that A is measured Eastward from North and H is measured Westward from South. For maintaining consistency across the problem, report your answers to two decimal places unless otherwise specified.

Part 1 (15 points): The Time Problem

a) (6 points) Calculate the difference (error) ΔT between the local sidereal time and the recorded time (clock time) for each star. Then, determine their mean (ΔT_{mean}) and standard deviation (σ_T).

b) (3 points) Calculate the mean of the time differences separately for the West stars (ΔT_W) and East stars (ΔT_E). Determine the discrepancy $\Delta T_W - \Delta T_E$.

c) (6 points) The astrolabe has a systematic zenith-distance error (dz). Evaluate the uncertainty on the time difference, $d(\Delta T)$, caused by the astrolabe. Show that by averaging the time differences of the East and West stars the uncertainty caused by the astrolabe can be reduced.

Part 2 (3 points): Atmospheric Refraction

Atmospheric refraction in radians is approximately given by

$$R = \frac{7.888 \times 10^{-5} P}{(273.0 + T) \tan(h)}$$

where P is measured in hPa, T in $^{\circ}\text{C}$, and h is the apparent altitude. Calculate the atmospheric refraction (R) in arcseconds and the true zenith distance (z_{true}).

Part 3 (15 points): Yersin's Lines of Position

a) (5 points) Yersin adopted an assumed latitude of his observatory to be $\phi_0 = 12^{\circ}15'00''$ N, and an assumed time error $\Delta T_0 = 50$ s. To determine how small errors in latitude ($d\phi$) and clock time (dT) affect the observed zenith distance, we start with the spherical law of cosines for the Pole-Zenith-Star (PZS) triangle:

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H \quad (1)$$

Taking the differential of this equation with respect to latitude ϕ and hour angle H (recognizing that $dH = -15 \cos \phi dT$ due to Earth's rotation), show that a change in latitude and time produces a shift in the calculated zenith distance given by:

$$dz = -\sin A \cos \phi dH - \cos A d\phi$$

Using the definition of the intercept $p = z_{\text{calc}} - z_{\text{true}} = -dz$ (z_{calc} is the calculated and z_{true} the true zenith distances) and defining spatial corrections $x = \cos \phi dH$ (East-West clock correction in arcseconds) and $y = d\phi$ (North-South latitude correction in arcseconds), the Line of Position (LOP) equation is defined as:

$$p = x \sin(A) + y \cos(A)$$

Hints: Taking the derivative of equation (1) gives:

$$-\sin z dz = (\cos \phi \sin \delta - \sin \phi \cos \delta \cos H) d\phi - (\cos \phi \cos \delta \sin H) dH$$

b) (10 points) After reducing the raw data, the following table is obtained.

Star	Azimuth (A)	Intercept (p)
A	285.0°	$-0.40''$
B	245.0°	$-18.77''$
C	70.0°	$+18.08''$
D	125.0°	$-8.15''$

Plot the four LOPs on (x, y) plane (mark your graph as "D02.3b"). Graphically determine the center of the common intersection region (x_0, y_0) .

Part 4 (17 points): Recovering the Location of Yersin Observatory and Error Analytics

- (4 points) Determine the true latitude of the observatory (ϕ_{true}) and the true time error (ΔT_{true}) using the time correction $dT_{\text{corr}} = x_0 / (15 \cos \phi_0)$ (in seconds). Assumed that $d\phi = \phi_{\text{true}} - \phi_0$.
- (8 points) Calculate the systematic error (dz) of the astrolabe (in arcseconds) using the East-West time difference; assuming that the average azimuth for the East stars is 90° and the West stars is 270° .
- (2 points) Calculate the observation random error $\varepsilon = |x_0 \sin(A) + y_0 \cos(A) - p_{\text{obs}}|$ for the four stars.
- (3 points) Calculate the position error d (in meters) between the observer's assumed coordinates and the calculated true position (x_0, y_0) . Assume $1''$ of angular distance corresponds to 31 m on the Earth's surface.

D03. The Mystery of "Little Red Dots" (50 points)

Recent observations from the James Webb Space Telescope (JWST) have uncovered a population of compact, red galaxies in the early universe, formally designated as "Little Red Dots" (LRDs). Their spectra imply the presence of Supermassive Black Holes (SMBHs) with masses that appear disproportionately large relative to their host galaxies. To independently verify the masses of these black holes, Tidal Disruption Events (TDEs) have been proposed as an observational tool. A TDE occurs when a star passes sufficiently close to a SMBH and is torn apart by tidal forces, producing a luminous flare as the debris is accreted. In this problem, you will analyze LRD observational data, deduce their scaling relations, evaluate the breakdown of standard dynamical models, and calculate parameters for a future cosmic survey mission.

Part 1 (28 points): Empirical Laws and Physical Significance

Assume the central SMBH mass (M_{BH}) and the total stellar mass (M_*^{tot}) of the host galaxy follow a scaling relation:

$$\log M_{\text{BH}} = \alpha + \beta \log M_*^{\text{tot}}$$

Observational data for 10 representative LRDs (masses expressed in $M_\odot$) are provided in Table D03.1.

Table D03.1. LRD data.

LRD ID	$\log M_*^{\text{tot}}$	$\Delta \log M_*^{\text{tot}}$	$\log M_{\text{BH}}$	$\Delta \log M_{\text{BH}}$
1	8.33	0.24	6.93	0.14
2	8.30	0.30	6.40	0.30
3	7.99	0.37	7.19	0.12
4	8.96	0.46	8.47	0.02
5	8.70	0.36	6.74	0.17
6	8.16	0.43	7.26	0.11
7	9.01	0.43	7.37	0.03
8	8.65	0.39	7.56	0.08
9	8.29	0.35	7.52	0.08
10	8.65	0.41	7.86	0.06

a) (5 points) Defining $x_i = \log M_{*,i}^{\text{tot}}$, $y_i = \log M_{\text{BH},i}$, and the simplified effective uncertainty ($\sigma_{\text{eff},i}$) for each data point as

$$\sigma_{\text{eff},i} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Fill the values of the statistical weights $w_i = 1/\sigma_{\text{eff},i}^2$ in your answer sheet.

b) (12 points) Plot 10 data points (x_i , y_i) with both horizontal and vertical error bars on graph paper (mark your graph as “D03.1b”).

c) (7 points) Apply the weighted least-squares (WLS) regression method using your calculated weights w_i to determine the central values and standard errors for the slope ($\beta \pm \Delta\beta$) and the intercept ($\alpha \pm \Delta\alpha$). Overlay the resulting best-fit line on your graph. For your calculation, you may rely on the standard WLS formulas provided below:

$$S_w = \sum w_i, \quad S_x = \sum w_i x_i, \quad S_y = \sum w_i y_i$$

$$S_{xx} = \sum w_i x_i^2, \quad S_{xy} = \sum w_i x_i y_i, \quad \Delta = S_w S_{xx} - (S_x)^2$$

$$\beta = (S_w S_{xy} - S_x S_y)/\Delta, \quad \alpha = (S_{xx} S_y - S_x S_{xy})/\Delta, \quad \Delta\alpha = \sqrt{S_{xx}/\Delta}, \quad \Delta\beta = \sqrt{S_w/\Delta}$$

(Note: Because this regression incorporates statistical weights (w_i), the resulting best-fit line may visually deviate from what you would expect from a standard, unweighted linear fit.)

d) (4 points) Select the most accurate answer for each question.

d.1) (2 point) The observed positive slope ($\beta > 0$) implies a coevolutionary trend, yet LRDs host black holes that are disproportionately massive compared to their host galaxies. What does this combination most likely suggest about the early universe?

- A. Black holes outpace their host galaxies (stellar component) initially, either by forming from heavy seeds or undergoing extreme early accretion.
- B. Galaxies fully assemble their stellar mass long before the central black hole begins its rapid growth.
- C. Star formation and black hole accretion are strictly locked at an identical rate throughout cosmic history.
- D. Dark matter halos suppress black hole growth until the host galaxy reaches a critical mass threshold.

d.2) (2 points) The data show a large scatter in stellar mass estimations ($\Delta \log M_*^{\text{tot}}$) for LRDs. What is the primary physical reason for this substantial uncertainty at high redshift ($z \approx 5$)?

- A. Interstellar dust significantly absorbs the black hole's X-ray emissions, altering the mass-to-light ratio.
- B. Rapid stellar collisions in the dense galactic core create transient luminous events that mimic a heavier stellar mass.
- C. The James Webb Space Telescope's UV sensors suffer from extreme thermal noise when observing at $z \approx 5$.
- D. The luminous accretion disk outshines the stellar continuum, making it difficult to isolate the host galaxy's light.

Part 2 (6 points): Standard Dynamical Model

Assume the spatial distribution of the host galaxy's stellar mass follows a Dehnen spherical density profile,

$$\rho(r) = \frac{(3 - \gamma)}{4\pi} M_*^{\text{tot}} \frac{a}{r^\gamma (r + a)^{4-\gamma}}$$

where a is the scale radius (a fundamental parameter marking the transition zone where the mass density profile shifts between its inner and outer logarithmic slopes) and γ defines the inner density slope ($0 < \gamma < 3$). The black hole's radius of influence (r_h) is defined as the radius where the enclosed stellar mass equals the black hole mass: $M_*(r_h) = M_{\text{BH}}$. The stellar component has an isotropic velocity dispersion $\sigma(r)$ and $M(r)$ represents the total enclosed mass within radius r .

a) (2 points) Consider a typical galaxy, applying the “isothermal core” assumption ($\sigma = \text{const}$) to the region $r_h \leq r \ll a$ leads to $\gamma = 2$. Therefore, the enclosed stellar mass is given by

$$M_*(r) = M_*^{\text{tot}} \left(\frac{r}{r+a} \right)$$

Derive the expression for r_h in terms of the scale radius a and the mass ratio $M_{\text{BH}}/M_*^{\text{tot}}$.

b) (4 points) Based on the result from a), calculate the ratio r_h/a for LRD ID 4 in Table D03.1. Why does the standard “isothermal core” assumption collapse for this object? Select the most accurate explanation:

- A. The ratio indicates the stellar mass vastly outweighs the black hole, rendering the Jeans equation inapplicable.
- B. The ratio is incredibly small, meaning the black hole has no gravitational influence on the core.
- C. The black hole's radius of influence extends deeply into the scale radius.
- D. The ratio proves that the scale radius a is shrinking over time due to dark matter interactions.

Part 3 (16 points): Destruction Physics and Data Barriers

a) (7 points) Consider a star of mass m_* and radius R_* approaching the SMBH at the distance $r \gg R_*$. The tidal force is defined as the difference between gravitational forces exerted by the SMBH at the star's center and at its surface. The tidal radius $r = r_t$ is determined by balancing the tidal force with the star's self-gravitational force at its surface. Derive r_t .

If r_t is smaller than the Schwarzschild radius ($R_S = 2GM_{\text{BH}}/c^2$), the star is swallowed intact rather than disrupted. The black hole mass at which $r_t = R_S$ is defined as the Hills mass, M_H . Formulate M_H in terms of m_* , R_* and calculate its numerical value (in $M_\odot$) for a Sun-like star.

b) (4 points) Using your results ($\alpha \pm \Delta\alpha, \beta \pm \Delta\beta$) from question c) in Part 1, determine the maximum stellar mass ($\log M_{*,\text{max}}^{\text{tot}}$) a galaxy can have while still potentially hosting an observable TDE of a Sun-like star (requiring $M_{\text{BH}} = M_H$). Then, calculate the range (Min to Max) of $\log M_{*,\text{max}}^{\text{tot}}$. Based on the extreme values, select the most accurate statement about the predictive reliability of this model:

- A. The model is highly reliable. The calculated range perfectly brackets the typical masses of observed galaxies hosting supermassive black holes.
- B. The model lacks predictive power. The lower bound is about the mass of a small star cluster, and the upper bound exceeds the mass of the observable universe.

C. The model fails entirely because a linear weighted least-squares regression cannot be mathematically applied to logarithmic astronomical data.

D. The model is reliable, but the unphysical lower bound indicates that we must manually adjust the scale radius to correct the data scatter.

c) (5 points) Cosmic Mission Design

Assume an all-sky cosmological population of $N \approx 1.021 \times 10^8$ LRDs located around redshift $z = 5$. The intrinsic (rest-frame) TDE rate per galaxy is estimated to be $\langle \dot{N}_{\text{TDE}} \rangle \approx 2.01 \times 10^{-6} \text{ yr}^{-1}$. Assume the number of detected TDEs from LRDs to be 5 during 1 year of continuous observation, what minimum fraction (in percentage) of the celestial sphere must the survey area cover?