

D01. Massive Neutron Stars and the Shapiro Time Delay (50 points)

The equation of state (EoS) of ultra-dense matter in neutron stars remains a major unsolved problem in fundamental physics. Theoretical models predicting a “soft” EoS dictate that neutron stars must collapse into black holes before reaching a mass of $2.0 M_{\odot}$. Consequently, a robust mass measurement of a neutron star exceeding $2.0 M_{\odot}$ would definitively rule out soft EoS models.

Radio observations of the millisecond pulsar PSR J0740+6620, located in a binary system with a white dwarf companion, offer a unique testbed for this prediction. According to General Relativity, as the pulsar passes through superior conjunction, when it is directly behind its companion, the companion's gravitational field curves spacetime, delaying the propagation of radio pulses to Earth. This phenomenon is known as the Shapiro time delay. Assuming an edge-on orbital inclination ($i \approx 90^\circ$) and a perfectly circular orbit (eccentricity $e = 0$), the Shapiro delay Δt_s is approximated by

$$\Delta t_s \approx \left(\frac{2GM_{WD}}{c^3} \right) [-\ln(1 - \cos \phi)] + b$$

where M_{WD} is the white dwarf mass, ϕ is the orbital phase ($\phi = 0^\circ$ at superior conjunction), and b is a baseline timing offset.

The binary system has an orbital period of $P_b = 4.767$ days. The pulsar has a projected semi-major axis of $x = 3.978$ light-seconds, which corresponds to an orbital radius $a_p = cx$. Assume the uncertainties on P_b and x are negligible.

The following table presents 15 observations of pulsar Shapiro delay at various orbital phases (ϕ) near superior conjunction. A constant measurement uncertainty of $\sigma_t = \pm 1.5 \mu s$ is assumed for all observations.

ID	Orbital Phase ϕ ($^\circ$)	Shapiro Delay Δt_s (μs)
1	2.0	18.5
2	-3.0	17.5
3	4.0	16.3
4	-5.0	13.9

ID	Orbital Phase ϕ (°)	Shapiro Delay Δt_s (μs)
5	6.0	12.8
6	8.0	12.5
7	10.0	10.1
8	−12.0	10.3
9	15.0	9.3
10	20.0	6.5
11	−22.0	5.8
12	25.0	6.6
13	30.0	5.0
14	−35.0	4.7
15	40.0	3.9

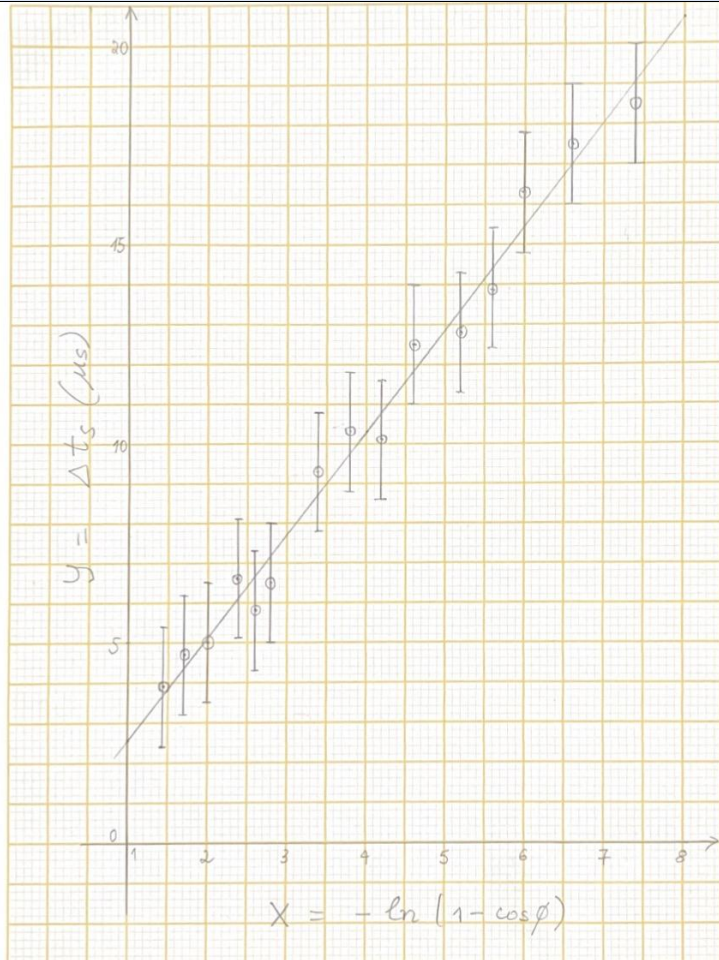
Part 1 (26 points): Estimation of White Dwarf Mass

a) (4 points) Define appropriate linearized variables (X and Y), and tabulate their values for 15 data points.

Solution				Point
Define appropriate linearized variables X and Y				1
$X = -\ln(1 - \cos \phi), \quad Y = \Delta t_s$				
Table:				
ID	ϕ (°)	X	Y (μs)	
1	2.0	7.40	18.5	
2	−3.0	6.59	17.5	
3	4.0	6.02	16.3	
4	−5.0	5.57	13.9	
5	6.0	5.21	12.8	
6	8.0	4.63	12.5	
7	10.0	4.19	10.1	
8	−12.0	3.82	10.3	
9	15.0	3.38	9.3	
10	20.0	2.81	6.5	

ID	ϕ (°)	X	Y (μs)	3
11	-22.0	2.62	5.8	
12	25.0	2.37	6.6	
13	30.0	2.01	5.0	
14	-35.0	1.71	4.7	
15	40.0	1.45	3.9	
Each correct row is awarded 0.2 points. Slight deviations due to rounding are acceptable.				

b) (18 points) Plot a scatter diagram of the 15 data points including their available error bars. Overlay the best-fit line on your graph (mark your graph as “D01.1b”). Determine the slope a , the intercept b , and their respective uncertainties (σ_a , σ_b).

Solution	Point
 Graph D01.1b.	

Graphing:

- Clear axis scaling and labels (X, Y). 1
- Plot correctly the 15 scatter points and their error bars ($\pm 1.5 \mu\text{s}$) 6
- Overlaid a reasonable best-fit line on the plot. 1

Summations:

$$\sum X = 59.78, \quad \sum Y = 153.70 \mu\text{s}$$

$$\sum X^2 = 287.05, \quad \sum XY = 738.28 \mu\text{s}$$

Coefficients:

Apply determinant formula:

$$\Delta = N \sum X^2 - \left(\sum X \right)^2 = 732.10$$

Slope:

$$a = \frac{N \sum XY - \sum X \sum Y}{\Delta} \approx 2.58 \mu\text{s} \quad 2$$

Intercept:

$$b = \frac{\sum X^2 \sum Y - \sum X \sum XY}{\Delta} \approx -0.02 \mu\text{s} \quad 2$$

Uncertainties:

Compute:

$$\sigma_a = \sigma_t \sqrt{\frac{N}{\Delta}} = 1.5 \sqrt{\frac{15}{732.10}} \approx 0.21 \mu\text{s} \quad 3$$

Compute:

$$\sigma_b = \sigma_t \sqrt{\frac{\sum X^2}{\Delta}} = 1.5 \sqrt{\frac{287.05}{732.10}} \approx 0.94 \mu\text{s} \quad 3$$

Results:

Using the formulas: $a = 2.58 \pm 0.21 \mu\text{s}$, $b = 0.0 \pm 0.9 \mu\text{s}$

Note: Students can determine a , b , along with their uncertainties, either by using the formulas for linear regression or from the graph.

Using the graph: $a = [2.4, 2.8]$ and $b = [-0.9, 0.9]$

$$\sigma_a = [0.1, 0.3] \text{ and } \sigma_b = [0.5, 1.5]$$

c) (4 points) Determine the mass of the white dwarf (M_{WD}) and its absolute uncertainty (ΔM_{WD}) in units of $M_{\odot}$.

Solution	Point
Mass expression: $M_{\text{WD}} = \frac{ac^3}{2G}$	1
Substituting the value of a into the equation gives: $M_{\text{WD}} \approx 5.21 \times 10^{29} \text{kg} \approx 0.262 M_{\odot}$	1
Error propagation expression: $\Delta M_{\text{WD}} = M_{\text{WD}} \frac{\sigma_a}{a}$	1
Final result: $\Delta M_{\text{WD}} \approx 0.021 M_{\odot}$	1
Conclusion: $M_{\text{WD}} = 0.262 \pm 0.021 M_{\odot}$	

Part 2 (13 points): Binary Dynamics and Error Propagation

a) (6 points) The mass function of the binary system is defined as

$$f_M = \frac{M_{\text{WD}}^3}{(M_{\text{PSR}} + M_{\text{WD}})^2}$$

where M_{PSR} is the mass of the pulsar. Derive the expression of f_M in terms of the observables a_p and P_b , and hence compute its value in units of $M_{\odot}$.

Solution	Point
Kepler's 3rd law:	

$P_b^2 = \frac{4\pi^2}{G(M_{\text{PSR}} + M_{\text{WD}})} (a_p + a_{\text{WD}})^3$	2
Center of mass momentum conservation:	
$a_{\text{WD}} = a_p \frac{M_{\text{PSR}}}{M_{\text{WD}}}$	2
Substitute a_{WD} into the above equation and isolate variables to derive the relation:	
$f_M = \frac{M_{\text{WD}}^3}{(M_{\text{PSR}} + M_{\text{WD}})^2} = \frac{4\pi^2 a_p^3}{G P_b^2} \quad (1)$	1
Computation:	
$f_M \approx 5.927 \times 10^{27} \text{ kg} \approx 0.002975 M_{\odot}$	1

b) (5 points) Determine the mass of the pulsar (M_{PSR}) and derive its uncertainty (ΔM_{PSR}).

Solution	Point
Rearrange (1) to solve for M_{PSR} :	
$M_{\text{PSR}} = \sqrt{\frac{M_{\text{WD}}^3}{f_M}} - M_{\text{WD}} \quad (2)$	1
Substitute values to obtain:	
$M_{\text{PSR}} \approx 2.20 M_{\odot}$	1
Differentiate both sides of (2) with respect to M_{WD} :	
$\frac{\Delta M_{\text{PSR}}}{\Delta M_{\text{WD}}} = \frac{3}{2} \sqrt{\frac{M_{\text{WD}}}{f_M}} - 1 \approx 13.08$	2
Error computation:	
$\Delta M_{\text{PSR}} \approx 13.08 \times \Delta M_{\text{WD}} \approx 0.27 M_{\odot}$	1
Conclusion:	

$M_{\text{PSR}} = 2.20 \pm 0.27 M_{\odot}$	
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c) (2 points) Determine how many standard deviations (σ) the measured pulsar mass lies above the upper mass limit predicted by the “soft EoS” model. Hence, state whether the model can be ruled out.

Solution	Point
Compute: $n_{\sigma} = \frac{M_{\text{PSR}} - M_{\text{limit}}}{\Delta M_{\text{PSR}}} = \frac{2.20 - 2.0}{0.27} = 0.74$	1
Answer: No <i>Since $n_{\sigma} < 3$, the measurement does not provide sufficient statistical significance to rule out the soft EoS theory.</i>	1

Part 3 (11 points): Pulse Time of Arrival (TOA): Effect of Multiple Factors

In binary pulsar timing, an observed Time of Arrival (TOA) is inherently shifted by various physical effects relative to an idealized baseline. To accurately determine the companion mass, astrophysicists must account for three primary components of the timing delay:

- Shapiro delay (Δt_{S}), caused by spacetime curvature around the white dwarf companion, produces a minuscule shift of $\sim 10 \mu\text{s}$.
- Roemer delay (Δt_{R}): The variation in pulse arrival time caused by the movement of the pulsar along its orbit around the binary's center of mass.
- Dispersion measure delay (Δt_{DM}): The propagation delay experienced by a signal traveling through the ionized interstellar medium. This delay is given by

$$\Delta t_{\text{DM}} = K \cdot DM \cdot \frac{1}{f^2}$$

where f is the frequency of the observed signal, K is the dispersion constant, and DM is the dispersion measure.

a) (3 points) Physical classification: Classify each of the following signal delay

mechanisms as achromatic (frequency-independent) or chromatic (frequency-dependent).

Roemer delay: ☐ Achromatic ☐ Chromatic

Dispersion measure delay: ☐ Achromatic ☐ Chromatic

Shapiro delay: ☐ Achromatic ☐ Chromatic

Solution	Point
Classification: <i>Achromatic:</i> Roemer delay, Shapiro delay <i>Chromatic:</i> Dispersion measure delay <i>(1 point for each correct answer)</i>	3

b) (2 points) To understand the difficulty of isolating the Shapiro delay, we must compare the magnitudes of these effects. Given an observing frequency $f = 1400$ MHz (standard L-band), an average interstellar dispersion measure $DM = 15$ pc cm⁻³, and the dispersion constant $K = 4.15 \times 10^3$ s MHz² pc⁻¹ cm³:

b.1) (1 points) Calculate Δt_{DM} .

Solution	Point
Substitute the values into the given formula: $\Delta t_{DM} = K \cdot DM \cdot \frac{1}{f^2}$ $\Delta t_{DM} = 4.15 \times 10^3 \times 15 \times \frac{1}{1400^2} \approx 0.032 \text{ s (32 ms)}$	1

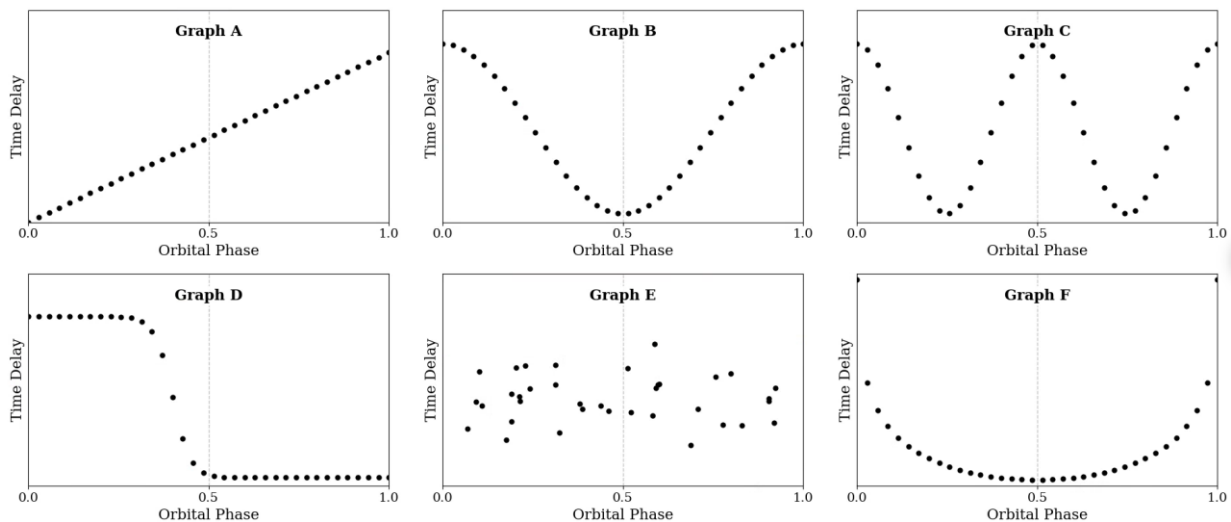
b.2) (1 points) Arrange the three delays (Δt_R , Δt_{DM} , Δt_S) in order from largest to smallest.

Solution	Point
Based on the problem text and their calculation, the correct order from largest to smallest is: Roemer delay (~ 4 s) > Dispersion measure delay	

(~ 32 ms) > Shapiro delay (~ 10 μ s).

1

c) (3 points) Below are six schematic graphs (A–F) illustrating the time delay as a function of orbital phase over one complete orbit. The vertical axis represents the delay amplitude (numerical values omitted). Match each of the three physical delay mechanisms listed below to the graph that best represents its expected shape.



Roemer delay [?]

Dispersion measure delay [?]

Shapiro delay [?]

Solution	Point
Roemer delay: B	1
Dispersion measure delay: E	1
Shapiro delay: F	1

d) (3 points) Observational strategy: Variations in the interstellar medium cause the dispersion measure to fluctuate over time, introducing timing noise. Select one correct answer to each question below.

d.1) (1 point) Why is a monochromatic observation (observing at a single frequency) insufficient to constrain the extremely small 10 μ s Shapiro delay?

- A) The telescope's receiver cannot process data faster than $10 \mu\text{s}$.
- B) Single frequencies are completely absorbed by the interstellar plasma.
- C) The system contains multiple sources of time delay, so the different physical effects cannot be distinguished from a single observation.
- D) Monochromatic signals do not experience gravitational time dilation.

Solution	Point
C	1

d.2) (2 points) Which observational strategy is used to measure and correct for dispersion measure variations?

- A) Integrating the signal over a one-year period to average out the dispersion measure variations.
- B) Conducting simultaneous observations at two distinct radio frequencies.
- C) Observing strictly at an extremely low frequency where the dispersion measure can be reduced.
- D) Only observing the pulsar when it is outside superior conjunction.

Solution	Point
B	2

D02. The Nha Trang Expedition (50 points)

In 1910, the scientist Alexandre Yersin (discoverer of the bubonic plague bacillus) established an astronomical observatory on the roof of his house in Nha Trang, Vietnam. There, he used a Claude–Driencourt prism astrolabe, designed to record the instant at which a star reaches a fixed zenith distance ($z = 30.0^\circ$). The following data table is inspired by Yersin's observations under atmospheric conditions of $T = 27.0^\circ\text{C}$ and $P = 1011.0 \text{ hPa}$.

$LST = 01^h39^m52.31^s - 02^h00^m30.38^s + 24^h = 23^h39^m21.93^s$ $\Delta T_C = 23^h39^m21.93^s - 23^h38^m31.61^s = \mathbf{50.32\ s}$ Star D: $LST = 01^h41^m38.84^s - 01^h37^m13.18^s = 00^h04^m25.66^s$ $\Delta T_D = 00^h04^m25.66^s - 00^h03^m35.22^s = \mathbf{50.44\ s}$ $\rightarrow \Delta T_{\text{mean}} = \frac{50.81 + 50.75 + 50.32 + 50.44}{4} = \mathbf{50.58\ s}$ $\rightarrow \sigma_T = \left(\frac{0.23^2 + 0.17^2 + 0.26^2 + 0.14^2}{4} \right)^{0.5} = \mathbf{0.21\ s}$	0.5
	0.5
	1
	1

b) (3 points) Calculate the mean of the time differences separately for the West stars (ΔT_W) and East stars (ΔT_E). Determine the discrepancy $\Delta T_W - \Delta T_E$.

Solution	Point
Stars A and B were observed in the West (setting). $\Delta T_W = \frac{\Delta T_A + \Delta T_B}{2} = \frac{50.81 + 50.75}{2} = \mathbf{50.78\ s}$	1
Stars C and D were observed in the East (rising). $\Delta T_E = \frac{\Delta T_C + \Delta T_D}{2} = \frac{50.32 + 50.44}{2} = \mathbf{50.38\ s}$	1
The discrepancy is: $\Delta T_W - \Delta T_E = 50.78 - 50.38 = \mathbf{0.40\ s}$	1

c) (6 points) The astrolabe has a systematic zenith-distance error (dz). Evaluate the uncertainty on the time difference, $d(\Delta T)$, caused by the astrolabe. Show that by averaging the time differences of the East and West stars the uncertainty caused by the astrolabe can be reduced.

Solution	Point
The time difference:	

$\Delta T = LST - T_R$	
The error on time measurement:	
$d(\Delta T) = d(LST) = d(\alpha + H) = dH$	1.5
Use the Sine-Cosine Rule for the triangle Pole-Zenith-Star (PZS), we have:	
$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H \quad (1)$	1.5
where ϕ is the latitude of the observatory and δ is the declination of the considered star.	
Differentiate equation (1):	
$-\sin z \, dz = -\cos \phi \cos \delta \sin H \, dH$	
Hence:	
$d(\Delta T) = dH = \frac{\sin z \, dz}{\cos \phi \cos \delta \sin H}$	1
For East stars:	
$H < 0 \rightarrow \sin H < 0$	0.5
For West stars:	
$H > 0 \rightarrow \sin H > 0$	0.5
Thus, the same zenith-distance error (dz) produces hour-angle corrections (dH) of the same magnitude but opposite sign. Averaging East and West observations cancels much of the systematic error caused by the astrolable:	
$\frac{1}{2} [d(\Delta T_W) + d(\Delta T_E)] < d(\Delta T_{W,E}) $	1

Part 2 (3 points): Atmospheric Refraction

Atmospheric refraction in radians is approximately given by

$$R = \frac{7.888 \times 10^{-5} P}{(273.0 + T) \tan(h)}$$

where P is measured in hPa, T in $^{\circ}\text{C}$, and h is the apparent altitude. Calculate the atmospheric refraction (R) in arcseconds and the true zenith distance (z_{true}).

Solution	Point
Atmospheric refraction in radians: $R = \frac{7.888 \times 10^{-5} \times 1011.0}{[(273.0 + 27.0) \times \tan 60^{\circ}]} = 1.5347 \times 10^{-4} \text{ rad}$	1
Convert to arcseconds: $R = 1.5347 \times 10^{-4} \times 206264.8 = \mathbf{31.66''}$	1
The true zenith distance: $z_{\text{true}} = 30^{\circ}00'00'' + 31.66'' = \mathbf{30^{\circ}00'31.66''}$ (The '+' sign because the atmospheric refraction acts like a lens, apparent altitude: $h_{\text{app}} = h_{\text{true}} + R \rightarrow (90 - z_{\text{app}}) = (90 - z_{\text{true}}) + R \rightarrow z_{\text{true}} = z_{\text{app}} + R$)	1

Part 3 (15 points): Yersin's Lines of Position

a) (5 points) Yersin adopted an assumed latitude of his observatory to be $\phi_0 = 12^{\circ}15'00''\text{N}$, and an assumed time error $\Delta T_0 = 50$ s. To determine how small errors in latitude ($d\phi$) and clock time (dT) affect the observed zenith distance, we start with the spherical law of cosines for the Pole-Zenith-Star (PZS) triangle:

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H \quad (1)$$

Taking the differential of this equation with respect to latitude ϕ and hour angle H (recognizing that $dH = -15 \cos \phi dT$ due to Earth's rotation), show that a change in latitude and time produces a shift in the calculated zenith distance given by:

$$dz = -\sin A \cos \phi dH - \cos A d\phi$$

Using the definition of the intercept $p = z_{\text{calc}} - z_{\text{true}} = -dz$ (z_{calc} is the calculated and z_{true} the true zenith distances) and defining spatial corrections $x = \cos \phi dH$

(East-West clock correction in arcseconds) and $y = d\phi$ (North-South latitude correction in arcseconds), the Line of Position (LOP) equation is defined as:

$$p = x \sin(A) + y \cos(A).$$

Hints: Taking the derivative of equation (1) gives:

$$-\sin z \, dz = (\cos \phi \sin \delta - \sin \phi \cos \delta \cos H) d\phi - (\cos \phi \cos \delta \sin H) dH$$

Solution	Point
Use the Law of Cosine for the triangle PZS: $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \cos A$ $\rightarrow \cos \phi \sin z \cos A = \sin \delta - \sin \phi \cos z$ Substitute $\cos z$ into the right side of the above equation: $\cos \phi \sin z \cos A = \sin \delta - \sin \phi (\sin \phi \sin \delta + \cos \phi \cos \delta \cos H)$ $\rightarrow \cos \phi \sin z \cos A = \sin \delta \cos^2 \phi - \sin \phi \cos \phi \cos \delta \cos H$ $\rightarrow \sin z \cos A = \cos \phi \sin \delta - \sin \phi \cos \delta \cos H \quad (2)$	1.5 1
From the Spherical Law of Sines: $\frac{\sin H}{\sin z} = -\frac{\sin A}{\cos \delta}$ Rearranging: $\sin z \sin A = -\cos \delta \sin H \quad (3)$ <i>(the minus sign because H is Westward and A is Eastward)</i> Substitute the identities (2) and (3) into the derivative of the zenith distance equation (in Hints): $-\sin z \, dz = (\sin z \cos A) d\phi + (\sin z \sin A) \cos \phi \, dH$ $dz = -\sin A (\cos \phi \, dH) - \cos A \, d\phi \quad \text{(Q.E.D)}$	1.5 1

b) (10 points) After reducing the raw data, the following table is obtained.

Star	Azimuth (A)	Intercept (p)
A	285.0°	−0.40''
B	245.0°	−18.77''
C	70.0°	+18.08''
D	125.0°	−8.15''

Plot the four LOPs on (x, y) plane (mark your graph as “D02.3b”). Graphically determine the center of the common intersection region (x_0, y_0) .

Solution	Point
Graph D02.3b Plotting: Draw the four LOPs $x \sin(A) + y \cos(A) = p$ using the supplied azimuths and intercepts which forms an intersection region around the center $(8.5'', 27'')$.	

Line A: (0, -2) (10,36)	1.5
Line B: (0,44) (21,0)	1.5
Line C: (0,53) (19,0)	1.5
Line D: (0,14) (-10,0)	1.5
x_0 and y_0 values are within the acceptable ranges: $7'' \leq x_0 \leq 10'', \quad 26'' \leq y_0 \leq 30''$	4

Part 4 (17 points): Recovering the Location of Yersin Observatory and Error Analytics

a) (4 points) Determine the true latitude of the observatory (ϕ_{true}) and the true time error (ΔT_{true}) using the time correction $dT_{\text{corr}} = x_0 / (15 \cos \phi_0)$ (in seconds). Assumed that $d\phi = \phi_{\text{true}} - \phi_0$.

Solution	Point
Latitude correction: $y_0 = d\phi = \phi_{\text{true}} - \phi_0$ $\rightarrow \phi_{\text{true}} = \phi_0 + y_0$ Therefore: $\phi_{\text{true}} = 12^\circ 15' 00'' + 27.20'' = \mathbf{12^\circ 15' 27.20''}$ Using $x_0 = 8.50''$ and $\phi_0 = 12^\circ 15'$ ($\cos \phi_0 \approx 0.977$) gives: $dT_{\text{corr}} = \frac{x_0}{15 \cos \phi_0} = \frac{8.50}{15 \times 0.977} = \mathbf{0.58 \text{ s}}$ Hence: $\Delta T_{\text{true}} = \Delta T_0 + dT_{\text{corr}} = 50.00 + 0.58 = \mathbf{50.58 \text{ s}}$	1 2 1

b) (8 points) Calculate the systematic error (dz) of the astrolabe (in arcseconds) using the East-West time difference; assuming that the average azimuth for the East stars is 90° and the West stars is 270° .

Solution	Point
From the result of question c) in Part 1, we have: $d(\Delta T) = \frac{\sin z \, dz}{\cos \phi \cos \delta \sin H}$ Using the Spherical Law of Sines: $\frac{\sin H}{\sin z} = -\frac{\sin A}{\cos \delta}$ $\rightarrow d(\Delta T) = -\frac{dz}{\cos \phi \sin A}$ For the Eastern star (the average azimuth $\sim 90^\circ$): $d(\Delta T_E) = -\frac{dz}{\cos \phi}$ For the Western stars (the average azimuth $\sim 270^\circ$): $d(\Delta T_W) = \frac{dz}{\cos \phi}$ Combining these results yields: $ d(\Delta T_W) - d(\Delta T_E) = \frac{2dz}{\cos \phi}$ Convert into seconds: $\Delta T_W - \Delta T_E = \frac{2dz}{15 \cos \phi}$ Therefore: $dz = \frac{15}{2} (\Delta T_W - \Delta T_E) \cos \phi_0 = \mathbf{2.93''}$	1 1 1 1 1 1 1 1

c) (2 points) Calculate the observation random error $\varepsilon = |x_0 \sin(A) + y_0 \cos(A) - p_{\text{obs}}|$ for the four stars.

Solution	Point
$\varepsilon_A = 8.50'' \sin(285.0^\circ) + 27.20'' \cos(285.0^\circ) + 0.40'' = \mathbf{0.77''}$	0.5
$\varepsilon_B = 8.50'' \sin(245.0^\circ) + 27.20'' \cos(245.0^\circ) + 18.77'' = \mathbf{0.43''}$	0.5
$\varepsilon_C = 8.50'' \sin(70.0^\circ) + 27.20'' \cos(70.0^\circ) - 18.08'' = \mathbf{0.79''}$	0.5
$\varepsilon_D = 8.50'' \sin(125.0^\circ) + 27.20'' \cos(125.0^\circ) + 8.15'' = \mathbf{0.49''}$	0.5

d) (3 points) Calculate the position error d (in meters) between the observer's assumed coordinates and the calculated true position (x_0, y_0) . Assume $1''$ of angular distance corresponds to 31 m on the Earth's surface.

Solution	Point
$d = \sqrt{x_0^2 + y_0^2} = \sqrt{8.50''^2 + 27.20''^2} = \mathbf{28.50''}$	2
$1''$ is roughly 31 m, therefore, $d = 28.50'' \times 31 \text{ m}/1'' \approx \mathbf{883 \text{ m.}}$	1

D03. The Mystery of “Little Red Dots” (50 points)

Recent observations from the James Webb Space Telescope (JWST) have uncovered a population of compact, red galaxies in the early universe, formally designated as “Little Red Dots” (LRDs). Their spectra imply the presence of Supermassive Black Holes (SMBHs) with masses that appear disproportionately large relative to their host galaxies. To independently verify the masses of these black holes, Tidal Disruption Events (TDEs) have been proposed as an observational tool. A TDE occurs when a star passes sufficiently close to a SMBH and is torn apart by tidal forces, producing a luminous flare as the debris is accreted. In this problem, you will analyze LRD observational data, deduce their scaling relations, evaluate the breakdown of standard dynamical models, and calculate parameters for a future cosmic survey mission.

Part 1 (28 points): Empirical Laws and Physical Significance

Assume the central SMBH mass (M_{BH}) and the total stellar mass (M_*^{tot}) of the host galaxy follow a scaling relation:

$$\log M_{\text{BH}} = \alpha + \beta \log M_*^{\text{tot}}$$

Observational data for 10 representative LRDs (masses expressed in $M_{\odot}$) are provided in Table D03.1.

Table D03.1. LRD data.

LRD ID	$\log M_{*}^{\text{tot}}$	$\Delta \log M_{*}^{\text{tot}}$	$\log M_{\text{BH}}$	$\Delta \log M_{\text{BH}}$
1	8.33	0.24	6.93	0.14
2	8.30	0.30	6.40	0.30
3	7.99	0.37	7.19	0.12
4	8.96	0.46	8.47	0.02
5	8.70	0.36	6.74	0.17
6	8.16	0.43	7.26	0.11
7	9.01	0.43	7.37	0.03
8	8.65	0.39	7.56	0.08
9	8.29	0.35	7.52	0.08
10	8.65	0.41	7.86	0.06

a) (5 points) Defining $x_i = \log M_{*,i}^{\text{tot}}$, $y_i = \log M_{\text{BH},i}$, and the simplified effective uncertainty ($\sigma_{\text{eff},i}$) for each data point as

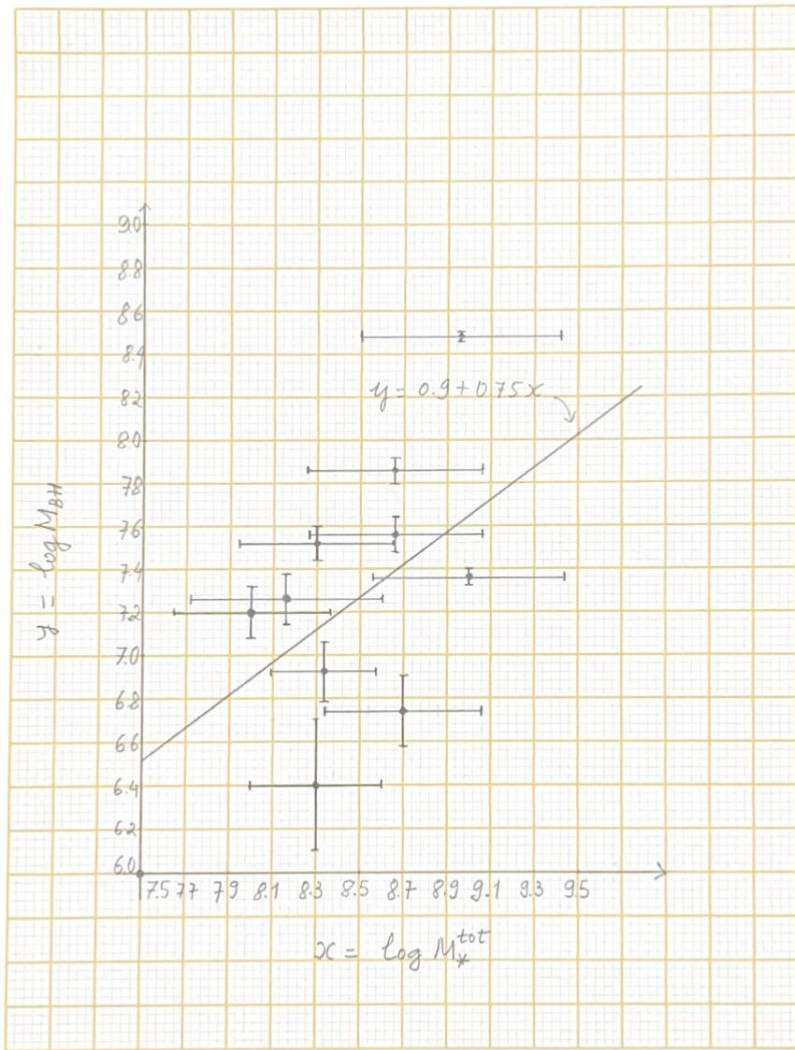
$$\sigma_{\text{eff},i} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Fill the values of the statistical weights $w_i = 1/\sigma_{\text{eff},i}^2$ in your answer sheet.

Solution				Point
LRD ID	$\Delta \log M_{*}^{\text{tot}} (\Delta x_i)$	$\Delta \log M_{\text{BH}} (\Delta y_i)$	Weight (w_i)	5
1	0.24	0.14	12.95	
2	0.30	0.30	5.56	
3	0.37	0.12	6.61	
4	0.46	0.02	4.72	
5	0.36	0.17	6.31	
6	0.43	0.11	5.08	
7	0.43	0.03	5.38	
8	0.39	0.08	6.31	
9	0.35	0.08	7.76	
10	0.41	0.06	5.82	

Note: 0.5 points for each corrected row.

b) (12 points) Plot 10 data points (x_i, y_i) with both horizontal and vertical error bars on graph paper (mark your graph as "D03.1b").

Solution	Point
Graphing:	
Clear axis scaling and labels (x, y) .	2
Plotting of the 10 scatter points (0.5 points per each).	5
Correct inclusion of error bars (both axes) for all points (0.5 points per each).	5
 Graph D03.1b	

c) (7 points) Apply the weighted least-squares (WLS) regression method using your calculated weights w_i to determine the central values and standard errors for the slope ($\beta \pm \Delta\beta$) and the intercept ($\alpha \pm \Delta\alpha$). Overlay the resulting best-fit line on your graph. For your calculation, you may rely on the standard WLS formulas provided below:

$$S_w = \sum w_i, \quad S_x = \sum w_i x_i, \quad S_y = \sum w_i y_i$$

$$S_{xx} = \sum w_i x_i^2, \quad S_{xy} = \sum w_i x_i y_i, \quad \Delta = S_w S_{xx} - (S_x)^2$$

$$\beta = \frac{S_w S_{xy} - S_x S_y}{\Delta}, \quad \alpha = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta}, \quad \Delta\alpha = \sqrt{\frac{S_{xx}}{\Delta}}, \quad \Delta\beta = \sqrt{\frac{S_w}{\Delta}}$$

(Note: Because this regression incorporates statistical weights (w_i), the resulting best-fit line may visually deviate from what you would expect from a standard, unweighted linear fit.)

Solution	Point
$S_w = \sum w_i \approx 66.49$	0.5
$S_x = \sum w_i x_i \approx 563.16$	0.5
$S_y = \sum w_i y_i \approx 483.65$	0.5
$S_{xx} = \sum w_i x_i^2 \approx 4775.63$	0.5
$S_{xy} = \sum w_i x_i y_i \approx 4100.80$	0.5
$\Delta = S_w S_{xx} - (S_x)^2 \approx 405.96$	0.5
<i>Note: Accept deviations in the final sums due to students truncating w_i floating points. Accept deviations up to ± 5.0 for large sums like S_{xx} and S_{xy}, and up to ± 0.5 for smaller sums like S_w and S_x.</i>	
$\alpha = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta} \approx 0.9$	0.5

$\beta = \frac{S_w S_{xy} - S_x S_y}{\Delta} \approx 0.75$	0.5
$\Delta\alpha = \sqrt{\frac{S_{xx}}{\Delta}} \approx 3.4$	0.5
$\Delta\beta = \sqrt{\frac{S_w}{\Delta}} \approx 0.40$	0.5
$\alpha = 0.9 \pm 3.4, \quad \beta = 0.75 \pm 0.40$	
<i>Note: If students report $\alpha = 1 \pm 3$ and $\beta = 0.8 \pm 0.4$, the marks remain unchanged.</i>	
Best-fit Line Plotting: Draw the line $y = 0.9 + 0.75x$ through the scatter plot.	2

d) (4 points) Select the most accurate answer for each question.

d.1) (2 points) The observed positive slope ($\beta > 0$) implies a coevolutionary trend, yet LRDs host black holes that are disproportionately massive compared to their host galaxies. What does this combination most likely suggest about the early universe?

A. Black holes outpace their host galaxies (stellar component) initially, either by forming from heavy seeds or undergoing extreme early accretion.

B. Galaxies fully assemble their stellar mass long before the central black hole begins its rapid growth.

C. Star formation and black hole accretion are strictly locked at an identical rate throughout cosmic history.

D. Dark matter halos suppress black hole growth until the host galaxy reaches a critical mass threshold.

Solution	Point
A	2

d.2) (2 points) The data show a large scatter in stellar mass estimations ($\Delta \log M_*^{\text{tot}}$) for LRDs. What is the primary physical reason for this substantial uncertainty at high redshift ($z \approx 5$)?

- A. Interstellar dust significantly absorbs the black hole's X-ray emissions, altering the mass-to-light ratio.
- B. Rapid stellar collisions in the dense galactic core create transient luminous events that mimic a heavier stellar mass.
- C. The James Webb Space Telescope's UV sensors suffer from extreme thermal noise when observing at $z \approx 5$.
- D. The luminous accretion disk outshines the stellar continuum, making it difficult to isolate the host galaxy's light.

Solution	Point
D	2

Part 2 (6 points): Standard Dynamical Model

Assume the spatial distribution of the host galaxy's stellar mass follows a Dehnen spherical density profile,

$$\rho(r) = \frac{(3 - \gamma)}{4\pi} M_*^{\text{tot}} \frac{a}{r^\gamma (r + a)^{4-\gamma}}$$

where a is the scale radius (a fundamental parameter marking the transition zone where the mass density profile shifts between its inner and outer logarithmic slopes) and γ defines the inner density slope ($0 < \gamma < 3$). The black hole's radius of influence (r_h) is defined as the radius where the enclosed stellar mass equals the black hole mass: $M_*(r_h) = M_{\text{BH}}$. The stellar component has an isotropic velocity dispersion $\sigma(r)$ and $M(r)$ represents the total enclosed mass within radius r .

a) (2 points) Consider a typical galaxy, applying the “isothermal core” assumption ($\sigma = \text{const}$) to the region $r_h \leq r \ll a$ leads to $\gamma = 2$. Therefore, the enclosed stellar mass is given by

$$M_*(r) = M_*^{\text{tot}} \left(\frac{r}{r + a} \right)$$

Solution	Point
The black hole's radius of influence r_h is the exact radius where the enclosed stellar mass equals the black hole mass. Set up the equality: $M_*(r_h) = M_{\text{BH}}$ $\Rightarrow M_*^{\text{tot}} \left(\frac{r_h}{r_h + a} \right) = M_{\text{BH}}$ Divide both sides by M_*^{tot} and isolate r_h: $\frac{r_h}{r_h + a} = \frac{M_{\text{BH}}}{M_*^{\text{tot}}}$ $r_h \left(1 - \frac{M_{\text{BH}}}{M_*^{\text{tot}}} \right) = a \frac{M_{\text{BH}}}{M_*^{\text{tot}}}$ Yield the derived expression: $r_h = a \frac{\frac{M_{\text{BH}}}{M_*^{\text{tot}}}}{1 - \frac{M_{\text{BH}}}{M_*^{\text{tot}}}}$	1 1

A. The ratio indicates the stellar mass vastly outweighs the black hole, rendering the Jeans equation inapplicable.

B. The ratio is incredibly small, meaning the black hole has no gravitational influence on the core.

C. The black hole's radius of influence extends deeply into the scale radius.

D. The ratio proves that the scale radius a is shrinking over time due to dark matter interactions.

Solution	Point
Ratio Calculation: For LRD 4 from the data table: $\log M_{*}^{\text{tot}} = 8.96$ and $\log M_{\text{BH}} = 8.47$. $\frac{M_{\text{BH}}}{M_{*}^{\text{tot}}} = \frac{10^{8.47}}{10^{8.96}} = 10^{-0.49} \approx 0.32$ Substitute this mass ratio into the relation derived in the previous part: $\frac{r_h}{a} \approx 0.48$ Physical Explanation (Multi-Choice Question): C Since $r_h \approx 0.48a$, the supermassive black hole's gravitational sphere of influence extends nearly halfway to the galaxy's scale radius. Unlike typical galaxies, LRDs do not exhibit an “intermediate region” ($r_h \ll r \ll a$).	1 1 2

Part 3 (16 points): Destruction Physics and Data Barriers

a) (7 points) Consider a star of mass m_* and radius R_* approaching the SMBH at the distance $r \gg R_*$. The tidal force is defined as the difference between gravitational forces exerted by the SMBH at the star's center and at its surface. The tidal radius $r = r_t$ is determined by balancing the tidal force with the star's self-gravitational force at its surface. Derive r_t .

If r_t is smaller than the Schwarzschild radius ($R_S = 2GM_{\text{BH}}/c^2$), the star is swallowed intact rather than disrupted. The black hole mass at which $r_t = R_S$ is defined as the Hills mass, M_H . Formulate M_H in terms of m_*, R_* and calculate its numerical value (in $M_\odot$) for a Sun-like star.

Solution	Point
Tidal force (ΔF): In a reference frame centered on the star, we evaluate the gravitational force at the center of the star: $F_{\text{center}} = \frac{GmM_{\text{BH}}}{r^2}$	0.5

where m is a test mass. At the near-side surface of the star (the point closest to the SMBH), located at a distance of $(r - R_*)$:

$$F_{\text{near}} = \frac{GmM_{\text{BH}}}{(r - R_*)^2} \quad 0.5$$

The tidal force is:

$$\Delta F = F_{\text{near}} - F_{\text{center}} = \frac{GmM_{\text{BH}}}{(r - R_*)^2} - \frac{GmM_{\text{BH}}}{r^2} \quad 1$$

$$\Delta F = \frac{GmM_{\text{BH}}}{r^2} \left[\frac{1}{\left(1 - \frac{R_*}{r}\right)^2} - 1 \right]$$

Using the first-order Taylor expansion for $R_*/r \ll 1$:

$$\left(1 - \frac{R_*}{r}\right)^{-2} \approx 1 + 2\left(\frac{R_*}{r}\right)$$

The tidal force is approximated by

$$\Delta F \approx \frac{2GmM_{\text{BH}}R_*}{r^3} \quad 1$$

At $r = r_t$, it balances with self-gravity:

$$\frac{2GmM_{\text{BH}}R_*}{r_t^3} = \frac{Gmm_*}{R_*^2} \quad 1$$

$$\Rightarrow r_t = R_* \left(\frac{2M_{\text{BH}}}{m_*} \right)^{\frac{1}{3}} \quad 1$$

When $r_t = R_S = 2GM_{\text{BH}}/c^2$, $M_{\text{BH}} = M_{\text{H}}$, and

$$M_{\text{H}} = \frac{1}{2} \left(\frac{c^2 R_*}{Gm_*^{1/3}} \right)^{3/2} \quad 1$$

For a Sun-like star, $R_* = R_{\odot}$, $m_* = M_{\odot}$, and

$$M_{\text{H}} \approx 1.62 \times 10^8 M_{\odot} \quad 1$$

b) (4 points) Using your results $(\alpha \pm \Delta\alpha, \beta \pm \Delta\beta)$ from question c) in Part 1, determine the maximum stellar mass ($\log M_{*,\max}^{\text{tot}}$) a galaxy can have while still potentially hosting an observable TDE of a Sun-like star (requiring $M_{\text{BH}} = M_{\text{H}}$). Then, calculate the range (Min to Max) of $\log M_{*,\max}^{\text{tot}}$. Based on the extreme values, select the most accurate statement about the predictive reliability of this model:

- A. The model is highly reliable. The calculated range perfectly brackets the typical masses of observed galaxies hosting supermassive black holes.
- B. The model lacks predictive power. The lower bound is about the mass of a small star cluster, and the upper bound exceeds the mass of the observable universe.
- C. The model fails entirely because a linear weighted least-squares regression cannot be mathematically applied to logarithmic astronomical data.
- D. The model is reliable, but the unphysical lower bound indicates that we must manually adjust the scale radius to correct the data scatter.

Solution	Point
The maximum stellar mass: $\log M_{*,\max}^{\text{tot}} = \frac{\log M_{\text{H}} - \alpha}{\beta}$ where $\alpha = 0.9 \pm 3.4$, $\beta = 0.75 \pm 0.40$, and $\log M_{\text{H}} = 8.21$ for a Sun-like star. $\log M_{*,\max}^{\text{tot}}$ reaches the maximum when $\beta = \beta_{\min} = 0.75 - 0.40 = 0.35$ and $\alpha = \alpha_{\min} = 0.9 - 3.4 = -2.5$. Result: $\log M_{*,\max}^{\text{tot}} = \frac{8.21 - (-2.5)}{0.35} \approx 31$ $\log M_{*,\max}^{\text{tot}}$ reaches the minimum when $\beta = \beta_{\max} = 0.75 + 0.40 = 1.15$ and $\alpha = \alpha_{\max} = 0.9 + 3.4 = 4.3$. Result: $\log M_{*,\max}^{\text{tot}} = \frac{8.21 - 4.3}{1.15} \approx 3.4$ Multi-Choice Question: B	1 1 1 1

Reason: The lower bound is about the mass of a small star cluster, which cannot host a SMBH. The upper bound largely exceeds the total mass of the entire observable universe. These derived boundaries have no physical reality.

c) (5 points) Cosmic Mission Design

Assume an all-sky cosmological population of $N \approx 1.021 \times 10^8$ LRDs located around redshift $z = 5$. The intrinsic (rest-frame) TDE rate per galaxy is estimated to be $\langle \dot{N}_{\text{TDE}} \rangle \approx 2.01 \times 10^{-6} \text{ yr}^{-1}$. Assume the number of detected TDEs from LRDs to be 5 during 1 year of continuous observation, what minimum fraction (in percentage) of the celestial sphere must the survey area cover?

Solution	Point
Identify the necessity to divide by $(1 + z)$ to convert the intrinsic rate to the observer rate on Earth: $\text{Rate}_{\text{obs}} = \frac{N \times \langle \dot{N}_{\text{TDE}} \rangle}{1 + z}$	2
Substitute $z = 5$ to calculate the correct all-sky rate: $\text{Rate}_{\text{obs}} = \frac{1.021 \times 10^8 \times 2.01 \times 10^{-6}}{1 + 5} \approx 34 \text{ events/year}$	1.5
Calculate the final percentage ratio: $\text{Area} = \frac{5}{34} = 15\% \text{ of the celestial sphere}$	1.5