

T01. Interstellar Object (10 points)

a) (5 points)

Marking Scheme	Point
$\frac{mV_{\infty}^2}{2} = \frac{mv_p^2}{2} - \frac{GmM_{\odot}}{r_p}$	1
$mpV_{\infty} = mr_pv_p$	1
$\frac{p^2V_{\infty}^2}{r_p^2} - \frac{2GM_{\odot}}{r_p} - V_{\infty}^2 = 0$	1
$r_p = \frac{p^2V_{\infty}^2}{GM_{\odot} + \sqrt{G^2M_{\odot}^2 + p^2V_{\infty}^4}}$	1
$r_p = 5.05 \times 10^{10} \text{ m}$	1

b) (2 points)

Marking Scheme	Point
$\left(\frac{L_{\odot}}{4\pi r_p^2}\right) \pi R_{\text{ISO}}^2 = 4\pi R_{\text{ISO}}^2 \sigma T^4$	1
$T = \left(\frac{L_{\odot}}{16\pi\sigma r_p^2}\right)^{\frac{1}{4}} = 4.8 \times 10^2 \text{ K}$	0.5
Yes	0.5

c) (3 points)

Marking Scheme	Point
$\Delta m = 2.5 \log_{10} \left(\frac{F_{\text{max}}}{F_{\text{min}}} \right)$	1
$\frac{F_{\text{max}}}{F_{\text{min}}} = \frac{a}{b}$	1

$\frac{a}{b} = 10^{\Delta m/2.5} = 6.3$	1
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T02. Oscillating Around a Lagrange Point (10 points)

a) (5 points)

Marking Scheme	Point
$\delta_1 m_1 = (R - \delta_1) m_2$	1
$\delta_1 \simeq \frac{m_2}{m_1} R$	1
$-\frac{Gm_1}{(R - \delta_2)^2} + \frac{Gm_2}{\delta_2^2} + \omega^2(R - \delta_1 - \delta_2) = 0$	1.5
$\delta_2 \simeq \left(\frac{m_2}{3m_1}\right)^{\frac{1}{3}} R$	1.5

b) (3 points)

Marking Scheme	Point
$\vec{F}_{\text{Coriolis}} = -2m_3(\vec{\omega} \times \vec{v}) = 0$ and $(\vec{F}_{\text{cent}})_z = -m_3[\vec{\omega} \times (\vec{\omega} \times \vec{r})]_z = 0$	1
$-Gm_3m_1 \frac{z}{\rho_1^3}$ and $-Gm_3m_2 \frac{z}{\rho_2^3}$	1
$\frac{d^2z}{dt^2} = -Gm_1 \frac{z}{\rho_1^3} - Gm_2 \frac{z}{\rho_2^3}$	1
Here: $\rho_1 = \sqrt{(R - \delta_2)^2 + z^2}$ and $\rho_2 = \sqrt{\delta_2^2 + z^2}$ <i>Note: The marks can be added up if the student omits intermediate steps, but the net force and motion equation are correct.</i>	

c) (2 points)

Marking Scheme	Point
$\frac{d^2z}{dt^2} + \omega_z^2 z = 0$ with $\omega_z = \sqrt{\frac{Gm_1}{(R - \delta_2)^3} + \frac{Gm_2}{\delta_2^3}} \simeq 2\omega$	1

$T = \frac{2\pi}{\omega_z} \simeq \frac{1}{2} T_{\text{Earth}} = 6 \text{ months}$ <i>Note: The student can directly substitute the mass and length values of the Earth-Sun system to ω_z to get $T = 6 \text{ months}$. The marks retain.</i>	1
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T03. Stellar Winds Launched Along Magnetic Field Lines (10 points)

a) (2 points)

Marking Scheme	Point
$\Omega_0 = \sqrt{\frac{GM}{r_0^3}}$	0.5
$\Phi_{\text{grav}}(r, z) = -\frac{GMm}{\sqrt{r^2 + z^2}}$	0.5
$\Phi_{\text{cent}}(r) = -\frac{1}{2} m \Omega_0^2 r^2$	0.5
$\Phi_{\text{eff}}(r, z) = -\frac{GMm}{\sqrt{r^2 + z^2}} - \frac{1}{2} m \Omega_0^2 r^2$	0.5

b) (5 points)

Marking Scheme	Point
$r = r_0(1 + x), \quad z \approx r_0 x \tan \theta_0$	0.5
$\Phi_{\text{grav}}(x) = -\frac{GMm}{r_0} [1 + 2x + x^2(1 + \tan^2 \theta_0)]^{-\frac{1}{2}}$	0.5
$\Phi_{\text{grav}}(x) \approx -m \Omega_0^2 r_0^2 \left[1 - x + x^2 \left(1 - \frac{\tan^2 \theta_0}{2} \right) \right]$	1.5
$\Phi_{\text{cent}}(x) = -m \Omega_0^2 r_0^2 \left(\frac{1}{2} + x + \frac{1}{2} x^2 \right)$	0.5
$\Phi_{\text{eff}}(x) = -m \Omega_0^2 r_0^2 \left[\frac{3}{2} + x^2 \left(\frac{3}{2} - \frac{\tan^2 \theta_0}{2} \right) \right]$	1
$\theta_0 < 60^\circ$	1

Note: Students can find $\theta_0 < 60^\circ$ from $F_{\text{cent}} > F_{\text{grav}}$, where F_{cent} and F_{grav} are centrifugal and gravitational forces, respectively.

c) (3 points)

Marking Scheme	Point
$\Delta L_w = \Delta M_w \Omega_0 r_A^2$	1
$\Delta L_{\text{acc}} = \Delta M_{\text{acc}} \Omega_0 r_0^2$	1
$\Delta L_w = \Delta L_{\text{acc}}$	1

T04. Great Dimming of Betelgeuse (10 points)

a) (2.5 points)

Marking Scheme			Point
$\Delta m = 0.75$			1
Half mark	Full mark	Half mark	
$0.65 \leq \Delta m < 0.70$	$0.70 \leq \Delta m \leq 0.80$	$0.80 < \Delta m \leq 0.85$	
$k_1 = \frac{F}{F_0} = 10^{-\frac{\Delta m}{2.5}}$			
$k_1 = 0.50$			0.5

b) (2.5 points)

Marking Scheme	Point
$F = 0.20F_0 + 0.80F_{\text{clump}}$	1.5
$k_2 = \frac{F_{\text{clump}}}{F_0} = 0.38$	1

c) (1.5 points)

Marking Scheme	Point
$F_{\text{clump}} = F_0 e^{-\tau}$	1

$\tau = 0.97$	0.5
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d) (3.5 points)

Marking Scheme	Point
$\tau = \rho\kappa\Delta z$	1
$V = 0.80\pi R^2\Delta z$	0.5
$M_{\text{dust}} = 0.80\pi R^2 \frac{\tau}{\kappa}$	1
$M_{\text{dust}} = 5.9 \times 10^{-10} M_{\odot}$	0.5
$M_{\text{eject}} = 1.2 \times 10^{-7} M_{\odot}$	0.5

T05. Primordial Black Holes (10 points)

a) (5 points)

Marking Scheme	Point
$\rho_{\text{PBH}} = \frac{M}{\frac{4}{3}\pi r_s^3}$	1
$r_s = \frac{2GM}{c^2}$	0.5
$\rho_{\text{PBH}} = \frac{3c^6}{32\pi G^3 M^2}$	0.5
$\rho_c \sim \rho_{\text{PBH}}$	2
$M \sim \frac{c^3 t}{6G}$	1
<u>Note:</u> - If students use "=", "≈", or "≤" for the density relation above, the marks remain unchanged. - Students can write the coefficient as $\sqrt{3/32\pi}$ or 1 instead of 1/6.	

b) (5 points)

Marking Scheme	Point
$\tau(M) \geq 13.8 \text{ (Gyr)}$	1.5
$M \geq 1.1 \times 10^{-19} \times M_{\odot}$	1
$\frac{c^3 t}{6G} \gtrsim 1.1 \times 10^{-19} \times M_{\odot}$	1.5
$t \gtrsim 3.3 \times 10^{-24} \text{ (s)}$	1
<u>Note:</u> • Deduct 0.5 points if students use either “$\tau(M) = 13.8 \text{ Gyr}$” or “$\tau(M) > 13.8 \text{ Gyr}$”. • The above comparisons are in the same order of magnitude. Therefore, coefficients are not important or can be flexibly varied.	

T06. Solar Neutrinos: From Experimental Measurement to Core Temperature Modeling (20 points)

a) (5 points)

Marking Scheme	Point
$n_{\text{H}_2\text{O}} = \frac{m}{M_{\text{H}_2\text{O}}} = \frac{5.0 \times 10^7}{18 \times 10^{-3}} \approx 2.8 \times 10^9 \text{ mol}$	0.5
$N_{\text{H}_2\text{O}} = n_{\text{H}_2\text{O}} \times N_A = (2.8 \times 10^9) \times (6.022 \times 10^{23})$ $\approx 1.7 \times 10^{33} \text{ molecules}$	0.5
$N_e = N_{\text{H}_2\text{O}} \times 10 = (1.7 \times 10^{33}) \times 10 = 1.7 \times 10^{34} \text{ electrons}$	1
$N_v = \Phi N_e \sigma \eta t$	2
$N_v = (5.1 \times 10^6) \times (1.7 \times 10^{34}) \times (1.0 \times 10^{-44}) \times 0.445 \times 86400$ $\approx 33 \text{ events}$	1

b) (6 points)

Marking Scheme	Point
$\gamma_{\min} = \frac{n}{\sqrt{n^2 - 1}}$	1
$K_{\text{Cherenkov}} = \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) m_e c^2 = \left(\frac{1.33}{\sqrt{1.33^2 - 1}} - 1 \right) \times 0.511 = 0.264 \text{ MeV}$	1
$K_{e,\max} = \frac{2E_v^2}{m_e c^2 + 2E_v}$	1
$K_{e,\max} = K_{\text{Cherenkov}}$	1
$2E_v^2 - 0.528E_v - 0.135 = 0$	1
$E_{v,\min} \approx 0.423 \text{ MeV}$	1

c) (9 points)

Marking Scheme	Point
$R + \Delta R = R(T + \Delta T) = R_0 e^{-B(T+\Delta T)^{-1/3}}$	1
$1 + \frac{\Delta R}{R} = e^{-BT^{-1/3} \left[\left(1 + \frac{\Delta T}{T}\right)^{-1/3} - 1 \right]}$	1
$\ln \left(1 + \frac{\Delta R}{R} \right) = -BT^{-1/3} \left[\left(1 + \frac{\Delta T}{T}\right)^{-1/3} - 1 \right]$	1
$\ln \left(1 + \frac{\Delta R}{R} \right) \approx \frac{\Delta R}{R}$	0.5
$\left[\left(1 + \frac{\Delta T}{T}\right)^{-1/3} - 1 \right] \approx \left(1 - \frac{1}{3} \frac{\Delta T}{T} \right) - 1 = -\frac{1}{3} \frac{\Delta T}{T}$	0.5
$\frac{\Delta R}{R} \approx -BT^{-1/3} \left(-\frac{1}{3} \frac{\Delta T}{T} \right) \Rightarrow \frac{\Delta R}{R} \approx \left(\frac{1}{3} BT^{-1/3} \right) \frac{\Delta T}{T} \Rightarrow \alpha = \frac{1}{3} BT^{-1/3}$	1

Note: If a student correctly determines α by taking the derivative of $R(T)$, give 5 points.

$$\begin{aligned}\frac{dR}{dT} &= R_0 \exp(-BT^{-1/3}) \cdot (-B) \cdot \left(-\frac{1}{3}\right) T^{-4/3} \\ &= R(T) \cdot \frac{B}{3} T^{-4/3} \quad (2 \text{ points})\end{aligned}$$

$$\Delta R \approx \frac{dR}{dT} \Delta T = R \cdot \frac{B}{3} T^{-4/3} \Delta T \quad (1 \text{ point})$$

$$\frac{\Delta R}{R} \approx \frac{B}{3} T^{-4/3} \Delta T = \frac{B}{3T^{1/3}} \cdot \frac{\Delta T}{T} \quad (1 \text{ point})$$

$$\alpha = \frac{1}{3} BT^{-1/3} \quad (1 \text{ point})$$

$$\frac{\Delta R}{R} = \frac{\Delta \Phi}{\Phi} = \frac{\Delta N_v}{N_v} = 2\% = 0.02$$

$$\alpha = \frac{1}{3} BT^{-1/3} = \frac{17757}{3 \times (1.5 \times 10^7)^{1/3}} \approx 24$$

$$\frac{\Delta T}{T} = \frac{\Delta R}{R} \frac{1}{\alpha} = \frac{0.02}{24} \approx 0.083\%$$

Note: If a student bypasses the algebraic linearization proof and directly substitutes the numerical constants and $\frac{\Delta R}{R} = 0.02$ into the original unapproximated exponential ratio equation to solve for $\frac{\Delta T}{T}$ numerically, award 4 points provided they correctly calculate $\frac{\Delta T}{T} \approx 0.083\%$.

T07. Zenith Blind Spot (15 points)

a) (4 points)

Marking Scheme	Point
$\frac{\sin(90 - \delta)}{\sin A} = -\frac{\sin z}{\sin H} \Rightarrow \sin A \sin z = -\sin H \cos \delta$	1

$\cos A \sin z \frac{dA}{dt} + (\sin A \cos z) \frac{dz}{dt} = -(\cos H \cos \delta) \frac{dH}{dt}$	1
At transit: $H = 0 \Rightarrow \cos H = 1$; $A = 0 \Rightarrow \cos A = 1$ and $\sin A = 0$	1
$\frac{dH}{dt} = \omega_{\oplus}$; $\sin z \frac{dA}{dt} = -\omega_{\oplus} \cos \delta \Rightarrow \omega_A = \frac{dA}{dt} = -\omega_{\oplus} \frac{\cos \delta}{\sin z}$	1

b) (4 points)

Marking Scheme	Point
$z_{\min} = \phi - \delta = 21.00^\circ - 21^\circ 08' 33'' = 513''$	1
$\omega_A = 1.0^\circ/\text{s}, \omega_{\oplus} = 15^\circ/\text{hr} = 0.00417^\circ/\text{s}.$	1
$z_{\text{limit}} \approx 802''$	1
Yes	1

c) (3 points)

Marking Scheme	Point
$L = 2\sqrt{z_{\text{limit}}^2 - z_{\min}^2} = 2\sqrt{802^2 - 513^2} = 1233''$	1
$v_{\text{star}} = \omega_{\oplus} \cos \delta = 15''/\text{s} \times \cos(21^\circ 08' 33'') = 13.99''/\text{s}$	1
$\Delta t = \frac{L}{v_{\text{star}}} = \frac{1233''}{13.99''/\text{s}} = 88 \text{ s}$	1

d) (4 points)

Marking Scheme	Point
$\sin z_{\text{limit}} \propto \cos \delta$	1
$\sin z_{\text{limit}} \propto \cos \phi$	1
$\phi_{\max} \sim 0^\circ$	1
$\phi_{\min} = \pm 90^\circ$	1

T08. Evolved Star Spirals (25 points)

Part 1 (10 points)

a) (2 points)

Marking Scheme	Point
$\Delta r_{\text{arm}}[\text{au}] = \Delta\theta[\text{arcsec}] \times d[\text{pc}] = 2.29 \times 1000 = 2290$	1
$b = \frac{\Delta r_{\text{arm}}}{2\pi} = \frac{2290}{2\pi} \approx 364 \text{ au rad}^{-1}$	1

b) (4 points)

Marking Scheme	Point
$r(t) = r_p + V_w t$	1
$\phi(t) = \frac{V_p}{r_p} t$	1
$r(t) = r_p + \frac{r_p V_w}{V_p} \phi(t)$	1
$r = r_p + \frac{r_p V_w}{V_p} \phi$	1

c) (2 points)

Marking Scheme	Point
$T = \frac{2\pi r_p}{V_p} = \frac{2\pi b}{V_w}$	1
$T = \frac{2\pi \times 364}{2.95} \approx 775 \text{ yr}$	1

d) (2 points)

Marking Scheme	Point
$d^3 = M_{\text{tot}} T^2$	1

$d = \sqrt[3]{M_{\text{tot}} T^2} = \sqrt[3]{6.6 \times 775^2} \approx 158 \text{ au}$	1
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Part 2 (15 points):

a) (6 points)

Marking Scheme	Point
$x = r \sin \theta \cos \phi = \frac{r_p V_w}{V_p} \phi \sin \theta \cos \phi$	1.5
$y = r \sin \theta \sin \phi = \frac{r_p V_w}{V_p} \phi \sin \theta \sin \phi$	1.5
$z = r \cos \theta = \frac{r_p V_w}{V_p} \phi \cos \theta$	1.5
$V_x = V_w \sin \theta \cos \phi$	0.5
$V_y = V_w \sin \theta \sin \phi$	0.5
$V_z = V_w \cos \theta$	0.5

b) (7 points)

Marking Scheme	Point
$V_{\text{los}} = V_y \sin i - V_z \cos i = V_w (\sin \theta \sin \phi \sin i - \cos \theta \cos i)$	2
$\sin \theta \sin \phi \sin i = \cos \theta \cos i$	1
$X = x = r \sin \theta \cos \phi$	1
$Y = y \cos i + z \sin i = r (\sin \theta \sin \phi \cos i + \cos \theta \sin i)$	1
$Y = r \sin \theta \sin \phi (\cos i + \tan i \sin i) = \frac{r \sin \theta \sin \phi}{\cos i}$	1
$\frac{X^2}{(r \sin \theta)^2} + \frac{Y^2}{\left(\frac{r \sin \theta}{\cos i}\right)^2} = 1$	1

c) (2 points)

Marking Scheme	Point
Axial ratio = $\frac{1}{\cos i}$	1
$\frac{1}{\cos i} = 1.1 \Rightarrow i \approx 24.6^\circ$	1

T09. Gravitational Lensing and Delay (20 points)

Part 1 (15 points)

a) (2 points)

Marking Scheme	Point
$z = \frac{\Delta\lambda}{\lambda} \simeq \frac{\Delta V}{c}$	0.5
$\Delta V = H_0 d$	0.5
$d_S = \frac{0.406c}{H_0} = 1800 \text{ Mpc}$	0.5
$d_L = \frac{0.042c}{H_0} = 190 \text{ Mpc}$	0.5
<i>Note: If students then calculate the angular diameter distance as $d/(1+z)$ they should also receive full points.</i>	

b) (3 points)

Marking Scheme	Point
$\theta d_S = 2\alpha d_{LS}$	1
$\theta d_L = b$	1
$\theta = \sqrt{\frac{2R^* d_{LS}}{d_L d_S}} = \sqrt{\frac{2R^* (d_S - d_L)}{d_L d_S}}$	1
<i>Note: If students include the minus sign in their answer, deduct 0.5 points.</i>	

c) (3 points)

Marking Scheme	Point
$b = 2.3 \text{ kpc}$	1
$R^* = \frac{2GM}{c^2}$	0.5
$M = \frac{R^* c^2}{2G} = \frac{\theta^2 d_L d_S}{2d_{LS}} \frac{c^2}{2G}$	0.5
$M = 1.6 \times 10^{11} M_\odot$	1

d) (7 points)

Marking Scheme	Point
$x = b' - b + \varepsilon$	1
$2\alpha' d_{LS} = \left(\theta' + \frac{\varepsilon}{d_L} \right) d_S$	0.5
$2\alpha' = \frac{2R^*}{\theta' d_L}$	0.5
$\frac{\theta'}{\theta} + \frac{\varepsilon}{b} = \frac{\theta}{\theta'}$ or $\frac{b'}{b} + \frac{\varepsilon}{b} = \frac{b}{b'}$	1
$x = \frac{\varepsilon}{2}$	2
$x' = \frac{\varepsilon}{2}$	1
$\Delta\theta = \frac{\varepsilon}{2d_L}$	1

Part 2 (5 points)

a) (2 points)

Marking Scheme	Point
$b = R_\odot$	0.5

$2\alpha = \frac{2R^*}{b} = \frac{2R^*}{R_\odot}$	0.5
$R^* = \frac{2GM_\odot}{c^2} = 2.95 \text{ km } (\sim 3 \text{ km})$	0.5
$2\alpha = \frac{2 \times 2.95}{6.96 \times 10^5} = 0.85 \times 10^{-5} [\text{rad}] = 1.75 [\text{arcsec}]$ <i>(1.75 – 1.78 arcsec for full mark)</i>	0.5

b) (3 points)

Marking Scheme	Point
$\delta = \int_{R_\odot}^{d_\odot} \frac{2\alpha}{c} dR$	1
$\delta = \int_{R_\odot}^{d_\odot} \frac{2R^*}{cR} dR = \frac{2R^*}{c} \int_{R_\odot}^{d_\odot} \frac{1}{R} dR$	0.5
$\delta = \frac{2R^*}{c} \ln\left(\frac{d_\odot}{R_\odot}\right)$	1
$\delta = \frac{2 \times 2.95}{3 \times 10^5} \ln\left(\frac{1.496 \times 10^8}{6.96 \times 10^5}\right) = 106 \mu\text{s}$	0.5
<i>Note: If students integrate from the solar radius $R_\odot$ to any $R' < 1 \text{ au}$, they will also receive full credit.</i>	

T10. The Chevallier-Polarski-Linder Parametrization for Dynamical Dark Energy (20 points)

a) (5 points)

Marking Scheme	Point
$\frac{d\rho_{\text{DE}}}{\rho_{\text{DE}}} = -3(1 + w_{\text{DE}}) \frac{da}{a}$	1
$\frac{d\rho_{\text{DE}}}{\rho_{\text{DE}}} = 3[1 + w_{\text{DE}}(z)] \frac{1}{1+z} dz$	1

$\int_{\rho_{DE,0}}^{\rho_{DE}} \frac{d\rho'_{DE}}{\rho'_{DE}} = \int_0^z \frac{3[1 + w_{DE}(z')]}{1 + z'} dz'$	1
$\ln\left(\frac{\rho_{DE}}{\rho_{DE,0}}\right) = \int_0^z \frac{3[1 + w_{DE}(z')]}{1 + z'} dz'$	1
$\rho_{DE}(z) = \rho_{DE,0} \exp\left[\int_0^z \frac{3[1 + w_{DE}(z')]}{1 + z'} dz'\right]$	1

b) (5 points)

Marking Scheme	Point
$\frac{d\rho_m}{\rho_m} = \frac{3}{1+z} dz$	1
$\int_{\rho_{m,0}}^{\rho_m} \frac{d\rho'_m}{\rho'_m} = \int_0^z \frac{3}{1+z'} dz'$	1
$\rho_m = \rho_{m,0}(1+z)^3$	1
$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2} \left\{ \rho_{m,0}(1+z)^3 + \rho_{DE,0} \exp \int_0^z \frac{3[1 + w_{DE}(z')]}{1 + z'} dz' \right\}$	1
$E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} \exp \left\{ \int_0^z \frac{3[1 + w_{DE}(z')]}{1 + z'} dz' \right\}$	1
<i>Note: Students can directly write down the formula of ρ_m without derivations. The marks remain unchanged.</i>	

c) (5 points)

Marking Scheme	Point
$E^2(a) = \Omega_{m,0}a^{-3} + \Omega_{DE,0} \exp \left\{ - \int_1^a \frac{3[1 + w_{DE}(a')]}{a'} da' \right\}$	1
$E^2(a) = \Omega_{m,0}a^{-3} + \Omega_{DE,0} e^{3w_a(a-1)} a^{-3[1+(w_0+w_a)]}$	3
$E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} e^{-\frac{3w_a z}{1+z}} (1+z)^{3[1+(w_0+w_a)]}$	1

Note: Students can directly compute $E(z)$ by substituting $a = 1/(1+z)$ into w_{DE} . The marks remain unchanged.

d) (5 points)

Marking Scheme	Point
$\Omega_{m,0}(1+z)^3 \gg \Omega_{DE,0} e^{-\frac{3w_a z}{1+z}} (1+z)^{3[1+(w_0+w_a)]}$	2
$\frac{\Omega_{m,0}(1+z)^3}{\Omega_{DE,0} e^{-3w_a}(1+z)^{3[1+(w_0+w_a)]}} \gg 1$	1
$\frac{(1+z)^3}{(1+z)^{3[1+(w_0+w_a)]}} \gg 1$	1
$w_0 + w_a < 0$	1

T11. Physics of Jupiter's Interior (50 points)

Part 1 (23 points):

a) (6 points)

Marking Scheme	Point
$dP = -\rho_z g dr$	2
$g = \frac{4}{3}\pi G \rho_z r$	1
$dP = -\frac{4}{3}\pi G \rho_z^2 r dr$	1
$P = -\frac{2}{3}\pi G \rho_z^2 r^2 + P_0$	1
$\rho_z = \frac{1}{R_c} \sqrt{\frac{3(P_0 - P_c)}{2\pi G}} = 1.43 \times 10^4 \text{ kg m}^{-3}$	1

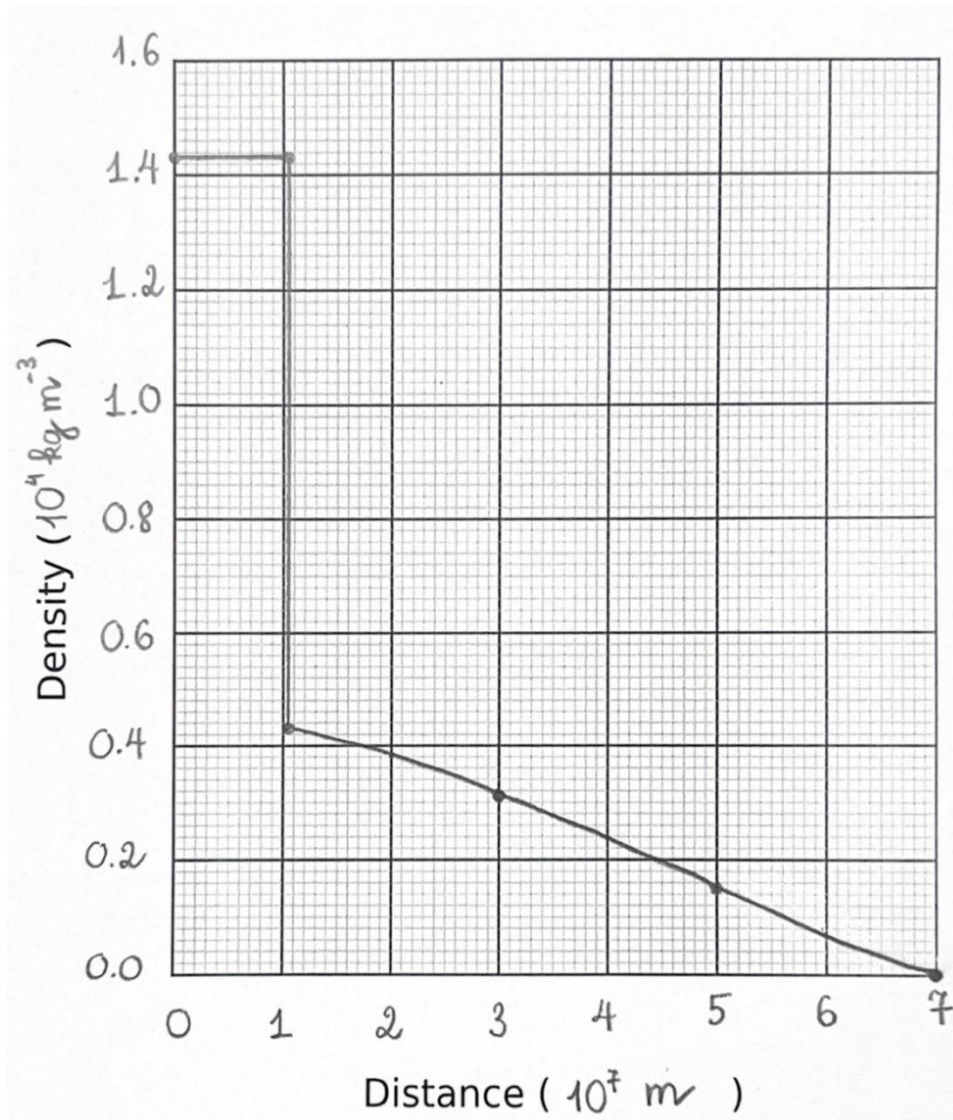
b) (6 points)

Marking Scheme	Point
$\rho_H(R_J) = \frac{A}{R_J} \cos\left(\frac{R_J}{R} + \varphi\right) = 0$	1
$\rho_H(R_C) = \frac{A}{R_C} \cos\left(\frac{R_C}{R} + \varphi\right) = \sqrt{\frac{P_C}{K}}$	1
$A = \frac{R_C}{\cos\left(\frac{R_C - R_J}{R} + \frac{\pi}{2}\right)} \sqrt{\frac{P_C}{K}}$	1
$\varphi = \frac{\pi}{2} - \frac{R_J}{R}$	1
$A = 9.46 \times 10^{10} \text{ kg m}^{-2}$	1
$\varphi = -1.54 \text{ rad}$	1
<i>Note: If the numerical value of the amplitude factor A is fall in a range of 9.46×10^{10} to $9.53 \times 10^{10} \text{ kg m}^{-2}$, the marks remain unchanged.</i>	

c) (8 points)

Solution	Point
The density profile	
• For the dense core:	
- Plot two critical points at $(0, \rho_Z)$ and (R_C, ρ_Z) .	1
- Plot a straight line passing through these points.	0.5
• For the fluid envelope:	
- Plot two critical points at $(R_C, \rho_H(R_C))$ and $(R_J, 0)$.	1
- Plot at least two additional points between $r = R_C$ and $r = R_J$ using $\rho_H = \frac{A}{r} \cos\left(\frac{r}{R} + \varphi\right)$.	1
- Plot an appropriate smooth curve passing through these points.	0.5

Note: It is optional to plot a straight line passing through two critical points (R_C, ρ_Z) and $(R_C, \rho_H(R_C))$.



The pressure profile

- For the dense core:

Plot two critical points at $(0, P_0)$ and (R_C, P_C) .

1

- For the fluid envelope:

- Plot one critical point at $(R_J, 0)$.

0.5

- Plot at least two additional points between $r = R_C$ and $r = R_J$ using $P = K\rho_H^2$.

1

• Graph finalization:

- Plot one more point in a range $0 \leq r \leq R_c$ [using Eq. (4)] or $R_c \leq r \leq R_j$ [using $P = K\rho_H^2$]

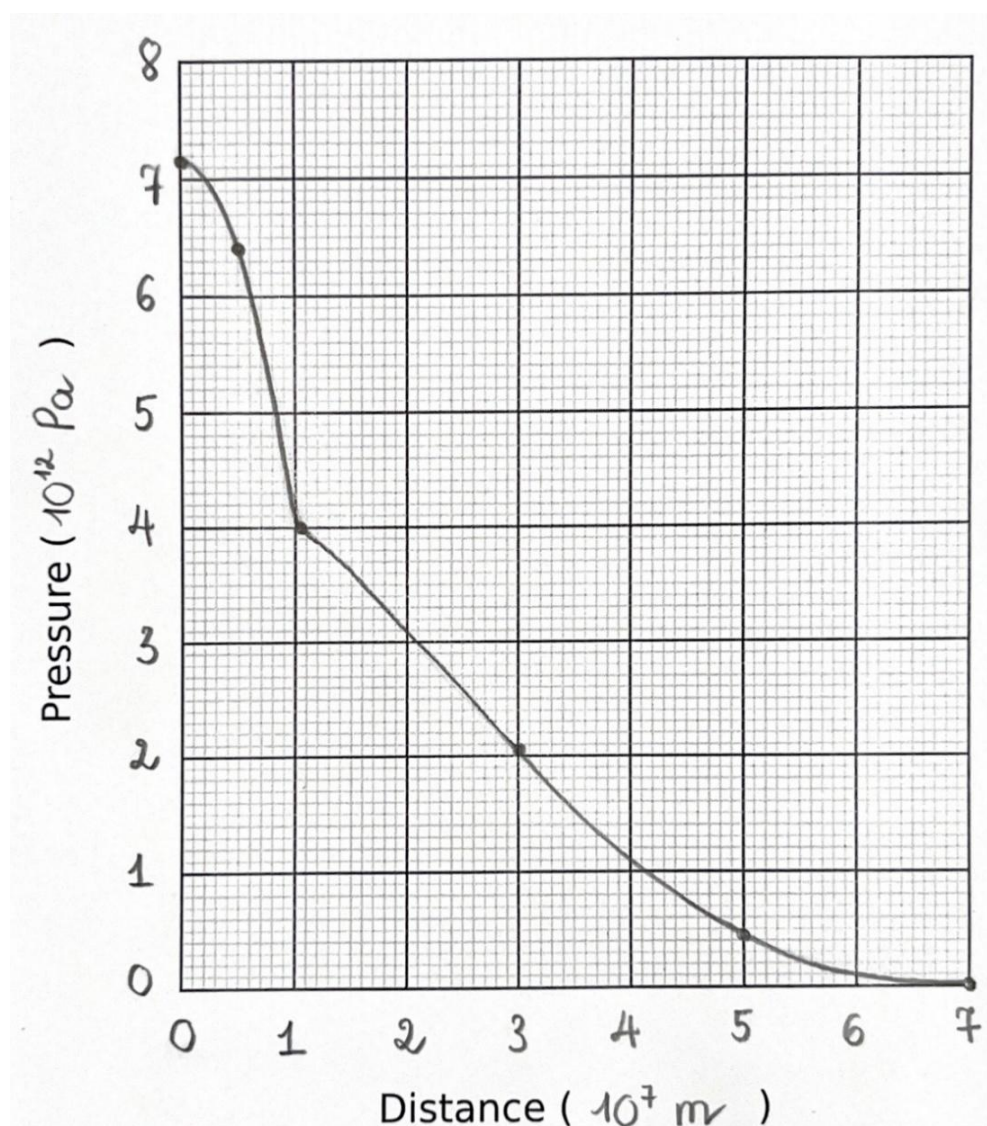
0.5

- Plot an appropriate smooth curve passing through all the points representing the dense core (parabola with vertex on y-axis).

0.5

- Plot an appropriate smooth curve passing through all the points representing the fluid envelope.

0.5



Note: Deduct 0.5 points for each graph missing axis units.

d) (3 points)

Marking Scheme	Point
$M_Z = \frac{4}{3}\pi R_C^3 \rho_Z = 6.93 \times 10^{25} \text{ kg}$	2
$M_H = M_J - M_Z = 1.83 \times 10^{27} \text{ kg}$	1

Part 2 (27 points):

a) (6 points)

Marking Scheme	Point
$dV = dv_Z + dv_H$	2
$dV = \frac{dM}{\rho}, \quad dv_Z = \frac{dm_Z}{\rho_Z}, \quad dv_H = \frac{dm_H}{\rho_H}$	1
$dm_Z = ZdM, \quad dm_H = (1 - Z)dM$	1
$\frac{1}{\rho} = \frac{Z}{\rho_Z} + \frac{1 - Z}{\rho_H}$	1
$\rho = \frac{\rho_Z \sqrt{P}}{Z\sqrt{P} + (1 - Z)\rho_Z \sqrt{K}}$	1

b) (7 points)

Marking Scheme	Point
$\rho = \frac{\rho_1 - \rho_2}{R_1 - R_2} r + \frac{\rho_2 R_1 - \rho_1 R_2}{R_1 - R_2}, \quad R_1 \leq r \leq R_2$	1
$\rho_1 = \frac{\rho_Z \sqrt{P_1}}{Z_1 \sqrt{P_1} + (1 - Z_1)\rho_Z \sqrt{K}} = 3.39 \times 10^3 \text{ kg m}^{-3}$	1
$\rho_2 = \frac{\rho_Z \sqrt{P_2}}{Z_2 \sqrt{P_2} + (1 - Z_2)\rho_Z \sqrt{K}} = 1.98 \times 10^3 \text{ kg m}^{-3}$	1
$M_{12} = \int_{R_1}^{R_2} 4\pi r^2 \rho dr$	1

$M_{12} = \frac{\pi}{3} (R_2 - R_1) [R_1^2 (3\rho_1 + \rho_2) + R_2^2 (\rho_1 + 3\rho_2) + 2R_1 R_2 (\rho_1 + \rho_2)]$	2
$M_{12} = 6.67 \times 10^{26} \text{ kg}$	1

c) (14 points)

Marking Scheme	Point
$M_1 + M_{12} + M_2 = M_J$	1
$Z_1 M_1 + M_{12}^{\text{rock}} + Z_2 M_2 = M_Z$	3
$M_1 = \frac{M_Z - M_{12}^{\text{rock}} - Z_2 (M_J - M_{12})}{Z_1 - Z_2}$	1
$M_2 = \frac{Z_1 (M_J - M_{12}) - M_Z + M_{12}^{\text{rock}}}{Z_1 - Z_2}$	1
$M_{12}^{\text{rock}} = \int_{R_1}^{R_2} 4\pi Z r^2 \rho dr$	1
$Z = \frac{Z_1 - Z_2}{R_1 - R_2} r + \frac{Z_2 R_1 - Z_1 R_2}{R_1 - R_2}, \quad R_1 \leq r \leq R_2$	1
$M_{12}^{\text{rock}} = \frac{\pi}{15} (R_2 - R_1) \{ [3Z_1 (4\rho_1 + \rho_2) + Z_2 (3\rho_1 + 2\rho_2)] R_1^2$ $+ [Z_1 (2\rho_1 + 3\rho_2) + 3Z_2 (\rho_1 + 4\rho_2)] R_2^2$ $+ 2(3Z_1 \rho_1 + 2Z_2 \rho_1 + 2Z_1 \rho_2 + 3Z_2 \rho_2) R_1 R_2 \}$	3
$M_{12}^{\text{rock}} = 2.84 \times 10^{25} \text{ kg}$	1
$M_1 = 3.83 \times 10^{26} \text{ kg}$	1
$M_2 = 8.50 \times 10^{26} \text{ kg}$	1

T12. Superluminous Galaxies in the Early Universe (50 points)

Part 1 (13 points)

a) (10 points)

Marking Scheme	Point
$E_{\text{obs}} = \frac{E_{\text{em}}}{1+z}$	2
$dt_{\text{obs}} = (1+z)dt_{\text{em}}$	2
$F_{\text{obs}} = \frac{L_{\text{obs}}}{4\pi D_M^2} = \frac{L}{4\pi D_M^2(1+z)^2}$	1
$D_L = D_M(1+z)$	1
$\theta = \frac{R(1+z)}{D_M}$	2
$D_A = \frac{D_M}{1+z}$	1
$D_L = D_A(1+z)^2$	1

b) (3 points)

Marking Scheme	Point
$\Sigma_{\text{obs}} = \frac{F_{\text{obs}}}{\Omega_{\text{obs}}} = \frac{\frac{L}{4\pi D_L^2}}{\frac{\pi R^2}{D_A^2}}$	1
$\Sigma_{\text{obs}} = \frac{L}{\pi R^2} \cdot \frac{D_A^2}{4\pi D_L^2} = \Sigma_{\text{int}} \cdot \frac{D_A^2}{4\pi D_L^2}$	1
$\Sigma_{\text{obs}} = \frac{\Sigma_{\text{int}}}{4\pi(1+z)^4}$	1

Part 2 (8 points)

Marking Scheme	Point
$F_{\text{opt,int}} = F_{\text{opt,obs}} \cdot 10^{\frac{A_V}{2.5}}$	1
$F_{\text{opt,int}} = 5.72 \times 10^{-19} \times 10^{\frac{2.79}{2.5}} = 7.47 \times 10^{-18} \text{ W m}^{-2}$	1

$F_{\text{bol}} = F_{\text{opt,int}} \cdot BC = 7.47 \times 10^{-18} \times 9.26 = 6.92 \times 10^{-17} \text{ W m}^{-2}$	1						
$\theta_{\text{rad}} = \frac{0.05}{3600} \times \frac{\pi}{180} = 2.424 \times 10^{-7} \text{ rad}$	1						
$\Omega_{\text{obs}} = \pi \theta_{\text{rad}}^2 = 1.846 \times 10^{-13} \text{ sr}$	1						
$\Sigma_{\text{obs_corr}} = \frac{F_{\text{bol}}}{\Omega_{\text{obs}}} = \frac{6.92 \times 10^{-17}}{1.846 \times 10^{-13}} = 3.75 \times 10^{-4} \text{ W m}^{-2} \text{ sr}^{-1}$	1						
$\Sigma_{\text{int_SI}} = 4 \times \pi \times 4655.0 \times 3.75 \times 10^{-4} = 21.94 \text{ W m}^{-2}$	1						
$\Sigma_{\text{int}} = \frac{21.94 \times 9.523 \times 10^{38}}{3.828 \times 10^{26}} = 5.46 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	1						
<table><tr><td>Half credit lower limit</td><td>Full credit range</td><td>Half credit upper limit</td></tr><tr><td>$5.35 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$</td><td>$5.40 \times 10^{13} \text{ to } 5.50 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$</td><td>$5.55 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$</td></tr></table>		Half credit lower limit	Full credit range	Half credit upper limit	$5.35 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.40 \times 10^{13} \text{ to } 5.50 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.55 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$
Half credit lower limit	Full credit range	Half credit upper limit					
$5.35 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.40 \times 10^{13} \text{ to } 5.50 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.55 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$					

Part 3 (18 points)

a) (6 points)

Marking Scheme	Point
$f = \sigma \frac{\Sigma_{\text{int_max}}}{c}$	2
$\sigma = m_{\text{d}} \kappa_{\text{dust}}$	1
$g_{\text{rad}} = \frac{f}{m_{\text{d}}} = \frac{\kappa_{\text{dust}} \Sigma_{\text{int_max}}}{c}$	1
$2\pi R^2 g_z = 4\pi G M_{\text{tot}}$	1
$g_z = 2\pi G \Sigma_{\text{tot}}$	1

b) (7 points)

Marking Scheme	Point
$g_{\text{rad}} = g_z$	1

$\Sigma_{\text{int_max}} = \frac{2\pi Gc\Sigma_{\text{tot}}}{\kappa_{\text{dust}}}$	1						
$\Sigma_{\text{Edd}} = \frac{\Sigma_{\text{int_max}}}{k}$	1						
$\Sigma_{\text{Edd}} = \frac{2\pi Gc\Sigma_{\text{tot}}}{k\kappa_{\text{dust}}}$	1						
$\Sigma_{\text{tot}} = \frac{2.3 \times 10^{10} \times 1.988 \times 10^{30}}{9.523 \times 10^{38}} = 48.01 \text{ kg m}^{-2}$	0.5						
$\Sigma_{\text{int_max}} = \frac{2\pi \times 6.674 \times 10^{-11} \times 2.998 \times 10^8 \times 48.01}{1.5} = 4.02 \text{ W m}^{-2}$	1						
$4.02 \text{ W m}^{-2} = 1.00 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	0.5						
$\Sigma_{\text{Edd}} = \frac{1.00 \times 10^{13}}{10^{10}} = 1000 \text{ M}_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$	1						
<table><tr><td>Half credit lower limit</td><td>Full credit range</td><td>Half credit upper limit</td></tr><tr><td>980</td><td>990 to 1010</td><td>1020</td></tr></table>		Half credit lower limit	Full credit range	Half credit upper limit	980	990 to 1010	1020
Half credit lower limit	Full credit range	Half credit upper limit					
980	990 to 1010	1020					

c) (5 points)

Marking Scheme	Point
$\Sigma_{\text{SFR_actual}} = \frac{\Sigma_{\text{int}}}{k}$	1
$\Sigma_{\text{SFR_actual}} = \frac{5.46 \times 10^{13}}{1.0 \times 10^{10}} = 5460 \text{ M}_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$	1
$\frac{\Sigma_{\text{SFR_actual}}}{\Sigma_{\text{Edd}}} = 5.46$	1
No	2

Part 4 (11 points)

a) (4 points)

Marking Scheme	Point
$M = \frac{v^2 R_{\text{em}}}{G}$	2
$M = 4.84 \times 10^8 M_{\odot}$	1
C	1

b) (7 points)

Marking Scheme	Point
$L_{\text{Edd}} = \frac{2\pi GMc}{\kappa_e}$	2
$\Sigma_{\text{Edd}} = \frac{2GMc}{\kappa_e R^2} \approx 37500 M_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$	2
$\frac{\Sigma_{\text{SFR_actual}}}{\Sigma_{\text{Edd}}} = \frac{5460}{37500} = 0.15$	1
Yes	2