

## INSTRUCTIONS

- You are given an envelope marked **B**. Please verify the Contestant Code printed on it. Inside you will find:
  - This Instruction Sheet in English and/or your native language (if applicable)
  - Data Sheet in English and/or your native language (if applicable)
  - A set of Working Sheets
  - One envelope marked **S** (please verify the Contestant Code marked on it).
- You are allowed to open the envelope **S** only when the whistle is blown to mark the start of the examination.**
- Inside the envelope **S** you will find the following:
  - Question Paper in English and/or your native language (if applicable)
  - A set of Summary Answer Sheets.
- Total duration of this examination is 5 hours.
- This examination consists of the following questions:

|               |                            |
|---------------|----------------------------|
| T01 – T05     | $5 \times 10$ points       |
| T06, T09, T10 | $3 \times 20 = 60$ points  |
| T07, T08      | $15 + 25 = 40$ points      |
| T11 – T12     | $2 \times 50 = 100$ points |
| Total:        | 250 points                 |

- Writing your answers:**
  - Summary Answer Sheet:
    - For each question a separate Summary Answer Sheet has been provided. Verify your Contestant Code on each page of the Summary Answer Sheet.
    - Final answer(s) for each question / each part of a question must be written in the corresponding box in the Summary Answer Sheet.
  - Working Sheets:
    - Your detailed work must be shown on the Working Sheet.

- Write your Contestant Code on the front page of each Working Sheet.
- Start a new question on a fresh Working Sheet. Write the question number in the box provided at the top.
- Write only on the front page of the Working Sheet (inside the boxed area). Do not write on the back page.
- For rough work that you do not want to be evaluated, clearly strike that part out.
- For each question, write the sheet number in the box provided at the top front page of the Working Sheet, starting with 1. For example, if you use 1 Working Sheet for T01, write "T01, 1" on that sheet. If you use 2 Working Sheets for T02, write "T02, 1" on the first sheet, and "T02, 2" on the second sheet. The total sheets used for each question must be filled.
- You may ask for extra Working Sheets, if required.
- **Note that failure to mark the Working Sheets, as instructed above, may result in your answers not being evaluated.**
- **At the end of the examination:**
  - Ensure your Contestant Code and sheet numbers are correctly written on all used Working Sheets.
  - Arrange your sheets in the following way:
  - Arrange the Summary Answer Sheets in numerical order from T01 to T12. All Summary Answer Sheets must be submitted, whether used or unused.
  - Arrange the Working Sheets in sequential order of the question numbers. Only include the Working Sheets that you actually used.
  - Place the bundle of Summary Answer Sheets on top of the bundle of Working Sheets.
  - Place all the sheets arranged in the above manner inside the envelope S.
  - **Only the contents of the envelope S will be evaluated. Any sheet that is not placed inside this envelope will not be evaluated.**
  - Place the envelope S inside the envelope B.
  - All other sheets, namely, Instruction Sheet(s), Data Sheet(s), Question Paper(s), and unused Working Sheet(s) should be placed directly inside the envelope B.
  - Tie the thread seal on the envelope B.
  - Hand over the envelope B to the examiner. Make sure that no sheet is left on the table, except the green and the red cards.
- **General Instructions:**
  - If you need a visit to the toilet during the examination, please raise your hand holding the green card kept on your table. You will be escorted by a volunteer.
  - If you need any other help during the examination, please raise your hand holding the red card kept on your table.
  - Use only black or blue pen for writing. For drawing, you may use a pencil.

- Some points may be deducted if the final answer is given without detailed solution.
- Some points may be deducted if the final answers have improper number of significant digits, no units or wrong units.
- The necessary values of universal constants and astronomical data should be taken from the Data Sheet provided to you.

### UNIVERSAL CONSTANTS

|                                  |                                                                                             |
|----------------------------------|---------------------------------------------------------------------------------------------|
| Avogadro constant                | $N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$                                               |
| Boltzmann constant               | $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$                                              |
| Electron charge magnitude        | $e = 1.602 \times 10^{-19} \text{ C}$                                                       |
| Speed of light in vacuum         | $c = 2.998 \times 10^8 \text{ m s}^{-1}$                                                    |
| Universal gravitational constant | $G = 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$                                   |
| Stefan-Boltzmann constant        | $\sigma = 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$                             |
| Permittivity of free space       | $\epsilon_0 = 8.854 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2$ |
| Mass of electron                 | $m_e = 9.109 \times 10^{-31} \text{ kg} = 0.511 \text{ MeV}/c^2$                            |
| Mass of proton                   | $m_p = 1.673 \times 10^{-27} \text{ kg} = 938.272 \text{ MeV}/c^2$                          |
| Mass of He nucleus               | $m_{\text{He}} = 6.645 \times 10^{-27} \text{ kg} = 3727.181 \text{ MeV}/c^2$               |

### ASTRONOMICAL DATA

|                            |                                                 |
|----------------------------|-------------------------------------------------|
| Mass of Sun                | $M_{\odot} = 1.988 \times 10^{30} \text{ kg}$   |
| Radius of Sun              | $R_{\odot} = 6.957 \times 10^8 \text{ m}$       |
| Luminosity of Sun          | $L_{\odot} = 3.828 \times 10^{26} \text{ W}$    |
| Distance from Earth to Sun | $d_{\odot} = 1.496 \times 10^{11} \text{ m}$    |
| Solar constant on Earth    | $F_{\odot} = 1361 \text{ W m}^{-2}$             |
| Hubble constant            | $H_0 = 67.7 \text{ km s}^{-1} \text{ Mpc}^{-1}$ |
| Age of the Universe today  | $t_0 = 13.8 \text{ Gyr}$                        |
| Astronomical Unit          | $1 \text{ au} = 1.496 \times 10^{11} \text{ m}$ |
| parsec                     | $1 \text{ pc} = 3.086 \times 10^{16} \text{ m}$ |
| light-year                 | $1 \text{ ly} = 9.461 \times 10^{15} \text{ m}$ |

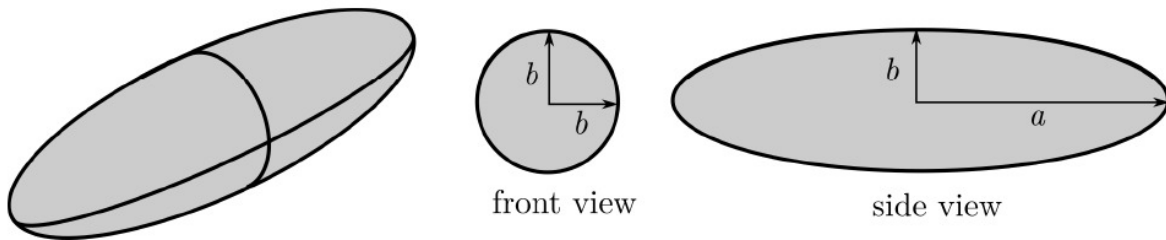
### MATHEMATICAL FORMULAS

|                                      |                                                                                                                                               |
|--------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------|
| Derivatives                          | $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$                                                |
| Integrals                            | $\int y dx = f$ so that $\frac{df}{dx} = y$ and<br>$\int_a^b y dx = f(b) - f(a)$                                                              |
| Chain rule                           | $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$                                                                                                 |
| Product rule                         | $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$                                                                                          |
| Quotient rule                        | $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(\frac{du}{dx}v - u\frac{dv}{dx}\right)$                                            |
| Powers                               | $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$ and $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + \text{constant}$<br>( $\alpha \neq -1$ ) |
| Exponents                            | $\frac{d}{dx}e^x = e^x$ and $\int e^x dx = e^x + \text{constant}$                                                                             |
| Logarithms                           | $\frac{d}{dx}\ln x  = \frac{1}{x}$ and $\int \frac{1}{x} dx = \ln x  + \text{constant}$                                                       |
| Sines                                | $\frac{d}{dx}(\sin x) = \cos x$ ,<br>$\int \sin x dx = -\cos x + \text{constant}$                                                             |
| Cosines                              | $\frac{d}{dx}(\cos x) = -\sin x$ ,<br>$\int \cos x dx = \sin x + \text{constant}$                                                             |
| Taylor expansions up to first orders | $(1+x)^\alpha = 1 + \alpha x$ for $ x  \ll 1$ ,<br>$(1+x_1)(1+x_2) = 1 + x_1 + x_2$ for $ x_1 ,  x_2  \ll 1$                                  |
| Taylor expansion up to second orders | $y = y_0 + y'_0(x - x_0) + \frac{1}{2}y''_0(x - x_0)^2$ for $x \approx x_0$                                                                   |

### T01. Interstellar Object (10 points)

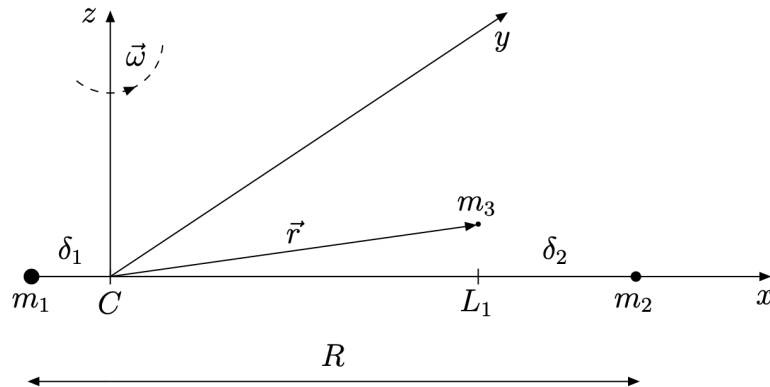
An interstellar object (ISO) enters the solar system on a hyperbolic orbit. The distance from the Sun to the asymptote of the ISO's trajectory (impact parameter) is  $p = 1.00$  au, and the ISO has an asymptotic speed  $V_\infty = 26 \text{ km s}^{-1}$  in the solar reference frame.

- (5 points) Determine the perihelion distance  $r_p$  of the ISO (in meters).
- (2 points) Calculate the equilibrium surface temperature  $T$  of the object at perihelion, assuming that the ISO is spherical, fast rotating, and has an ideal blackbody surface composed of water ice. In space the water ice sublimates at 120 K. Do we expect the water ice on ISO's surface to sublimate? Write YES / NO as the final answer.
- (3 points) Observations show that the object's apparent magnitude varies periodically with a light-curve peak-to-peak amplitude of 2.0 mag due to its spinning, revealing the non-spherical shape of the ISO. Assume that the object is an ellipsoid and the flux is proportional to the projected area during rotation and the ISO is rotating about its minor axis, which is perpendicular to the line-of-sight. Estimate the ratio of the major axis to minor axis of the object,  $a/b$ .



### T02. Oscillating Around a Lagrange Point (10 points)

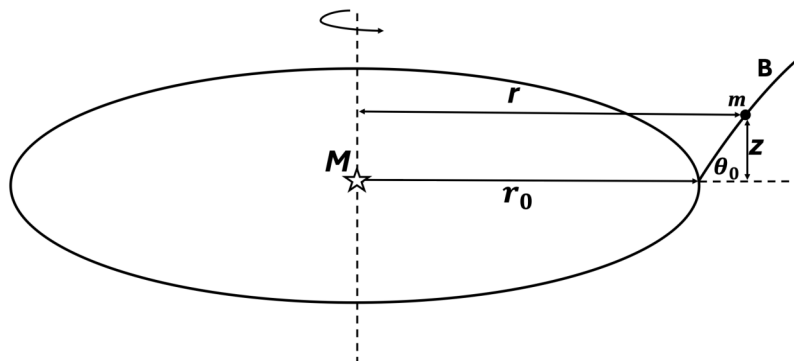
A binary system, whose masses  $m_1$  and  $m_2$  are hierarchical,  $m_1 \gg m_2$ , and separated by a distance  $R$ , circularly orbits its center of mass,  $C$ . The rotating reference frame  $Cxyz$  has the  $x$ -axis pointing from  $m_1$  to  $m_2$  and the  $z$ -axis oriented with the angular velocity of the system,  $\vec{\omega}$ . A satellite whose mass  $m_3$  is much smaller than  $m_1, m_2$  oscillates in the  $z$ -direction about the Lagrange point  $L_1$  lying on the line segment that connects  $m_1$  and  $m_2$ . Let  $\delta_1$  and  $\delta_2$  be the distances from  $C$  to  $m_1$  and  $L_1$  to  $m_2$ , respectively.



- (5 points) Determine  $\delta_1$  and  $\delta_2$  as functions of  $m_1$ ,  $m_2$ , and  $R$  at the leading order.
- (3 points) Find the equation of motion of the satellite along the  $z$ -axis.
- (2 points) Estimate the period  $T$  of small oscillations of the satellite in the  $z$ -direction for the Earth-Sun system.

### T03. Stellar Winds Launched Along Magnetic Field Lines (10 points)

Angular momentum from a circular Keplerian accretion disk around a young star of mass  $M$  is removed by stellar winds. Wind particles slide frictionlessly along magnetic field lines corotating with the disk. Consider a wind particle of mass  $m$  at radius  $r$  and height  $z$  moving along a magnetic field line that corotates with the disk at radius  $r_0$  and is inclined at an angle  $\theta_0$ .



- (2 points) In the frame corotating with the particle, derive  $\Phi_{\text{eff}}(r, z)$ , defined as the potential energy from the star's gravity and centrifugal force.
- (5 points) Introducing the variable  $x = \frac{r-r_0}{r_0}$ , write  $\Phi_{\text{eff}}$  in the vicinity of  $r_0$  to second-order approximation in  $x$ . Find the inclination angle  $\theta_0$  so that the particle can be pulled away from the disk. Hint:  $(1+u)^{-1/2} \approx 1 - \frac{u}{2} + \frac{3}{8}u^2$  with  $|u| \ll 1$ .

c) (3 points) At radius  $r = r_A$ , particles begin to decouple from magnetic field. The wind launched from  $r_0$  carries mass-loss rate  $\Delta M_w$  while the accretion rate is  $\Delta M_{acc}$ . If all accretion angular momentum is removed by the wind, show that  $\frac{\Delta M_{acc}}{\Delta M_w} = \left( \frac{r_A}{r_0} \right)^2$ .

#### T04. Great Dimming of Betelgeuse (10 points)

Between late 2019 and early 2020, the red supergiant Betelgeuse experienced an unusual visual fading event known as the “Great Dimming” (Figure T04.1). One explanation is that the star ejected material that formed a dust clump, obscuring the stellar disk. Assume the dimming occurred at a normal pulsation minimum and was caused entirely by the cylindrical dust clump covering 80% of the stellar disk.

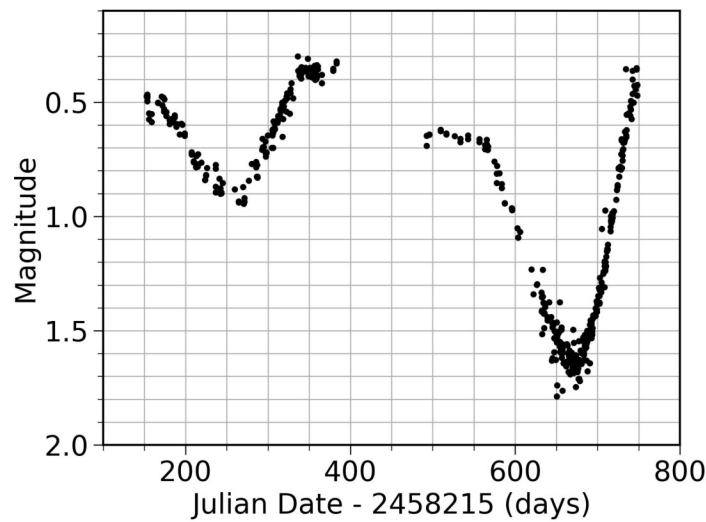


**Figure T04.1.** Visual light curve of Betelgeuse.

- (2.5 points) Estimate the ratio ( $k_1$ ) of the observed brightness at the minimum light to that at the normal pulsation minimum.
- (2.5 points) Estimate the fraction ( $k_2$ ) of light transmitted through the obscured region.
- (1.5 points) Calculate the optical depth  $\tau$  of the dust clump.
- (3.5 points) Assuming a mass absorption coefficient of the dust  $\kappa = 1000. \text{ m}^2 \text{ kg}^{-1}$ , a gas-to-dust mass ratio of 200., and a radius of Betelgeuse of  $1000 R_\odot$ , estimate the total mass ( $M_{\text{eject}}$ ) of the ejected material in units of  $M_\odot$ .

#### T05. Primordial Black Holes (10 points)

A primordial black hole (PBH) is a hypothetical black hole, which is expected to be formed very soon after the Big Bang.

- a) (5 points) Find the expression of the initial mass  $M$  of a PBH, provided that the cosmological density at a time  $t$  after the Big Bang is  $\rho_c \sim 1/(Gt^2)$  and the PBH is assumed to be a homogeneous sphere having a radius identical to the Schwarzschild radius.
- b) (5 points) Estimate the initial mass and formation time of PBHs after the Big Bang such that they are still evaporating, provided that the black hole evaporation time is  $\tau(M) = 10^{67} (M/M_\odot)^3$  (yr).

**T06. Solar Neutrinos: From Experimental Measurement to Core Temperature Modeling (20 points)**

High-energy solar neutrinos from the pp-III chain are detected by Earth-based experiments like Super-Kamiokande to study the Sun's core properties.

- a) (5 points) The pp-III neutrino flux on Earth is  $\Phi = 5.1 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$ . The Super-Kamiokande detector is a tank containing  $5.0 \times 10^7 \text{ kg}$  of pure water ( $\text{H}_2\text{O}$ ). The molar mass of  $\text{H}_2\text{O}$  is 18 g/mol. Its detection efficiency is  $\eta = 44.5\%$ . Assume that solar neutrinos only interact with the detector through elastic scattering with electrons. The average neutrino-electron interaction cross-section is  $\sigma \approx 1.0 \times 10^{-44} \text{ cm}^2$  in water. Physically,  $\sigma$  can be understood as the effective target area a neutrino sees when approaching an electron; a larger  $\sigma$  means a higher probability of collision. Calculate the number of events captured in a single day  $N_\nu$  (24 hours).
- b) (6 points) Neutrinos ( $\nu$ ) are detected indirectly via Cherenkov radiation emitted by scattered electrons ( $e^-$ ), i.e., a product of the electron-neutrino scattering reaction ( $\nu + e^- \rightarrow e^- + \nu$ ), in water (refractive index  $n = 1.33$ ). This event occurs when the velocity of electron satisfies  $v \geq c/n$ . Hence, we must use relativistic mechanics. The relativistic kinetic energy of electron is given as  $K = (\gamma - 1)m_e c^2$ , where the Lorentz factor  $\gamma$  is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The post-scattering energy of the neutrino ( $E'_\nu$ ) is related to its pre-scattering counterpart ( $E_\nu$ ) as

$$E'_\nu = \frac{E_\nu}{1 + \frac{E_\nu}{m_e c^2} (1 - \cos \theta)}$$

where  $\theta$  is the scattering angle of the neutrino. Assuming the electron's initial kinetic and binding energies are negligible, determine the minimum initial neutrino energy ( $E_{\nu, \min}$ ) in MeV required to generate a Cherenkov radiation event.

c) (9 points) The solar pp-III reaction rate  $R(T)$  depends on the core temperature  $T$  via the Gamow equation  $R(T) = R_0 e^{-BT^{-\frac{1}{3}}}$ , where  $R_0$  is a constant,  $B = 17757 \text{ K}^{1/3}$  is the Gamow factor, and  $T = 1.5 \times 10^7 \text{ K}$  is the steady-state core temperature.

Suppose the temperature fluctuates slightly ( $T \rightarrow T + \Delta T$ ), causing a small fractional reaction rate change ( $R \rightarrow R + \Delta R$ ). The relation between the fractional change in the reaction rate and temperature is in the form  $\frac{\Delta R}{R} = \alpha \frac{\Delta T}{T}$ , where  $\alpha$  is the thermal sensitivity coefficient. Express  $\alpha$  in terms of  $B$  and  $T$ . If the number of events detected by the Super-Kamiokande detector in a single day increases by 2%, calculate the corresponding percentage change in the Sun's core temperature.

Hints:  $(1 + x)^\delta \approx 1 + \delta x$  and  $\ln(1 + x) \approx x$  for  $|x| \ll 1$ .

### T07. Zenith Blind Spot (15 points)

An observer in Hanoi (latitude of exactly  $\phi = 21^\circ \text{ N}$ ) uses an altazimuth telescope to observe Zeta Tauri (declination  $\delta = +21^\circ 08' 33''$ ) with a maximum azimuthal tracking speed of  $1.0^\circ/\text{s}$ . As an object transits close to the zenith, the required tracking angular velocity increases sharply. If the required angular velocity exceeds the motor's maximum speed, the telescope loses its target. This problem is known as the "Zenith blind spot" or the "Gimbal lock". The position of the star is defined by its azimuth ( $A$ ), zenith distance ( $z$ ), and hour angle ( $H$ ). Note that  $A$  is measured Eastward from North and  $H$  is measured Westward from South.

a) (4 points) Show that for the telescope to be able to track the star, the required azimuthal angular velocity at meridian transit is given by  $\omega_A = -\omega_\oplus \frac{\cos \delta}{\sin z}$  where  $\omega_\oplus = 15^\circ/\text{hr}$  is the Earth's sidereal rotation rate.

b) (4 points) Calculate the minimum zenith distance ( $z_{\min}$ ) reached by Zeta Tauri during its transit and the radius of the blind spot ( $z_{\text{limit}}$ ) in arcseconds. Show that Zeta Tauri falls into the blind spot of the telescope during its transit over Hanoi.

c) (3 points) Assuming the star's path across the sky near the zenith can be approximated as a straight line, calculate the total time  $\Delta t$  (in seconds) that Zeta Tauri spends inside the telescope's blind spot.

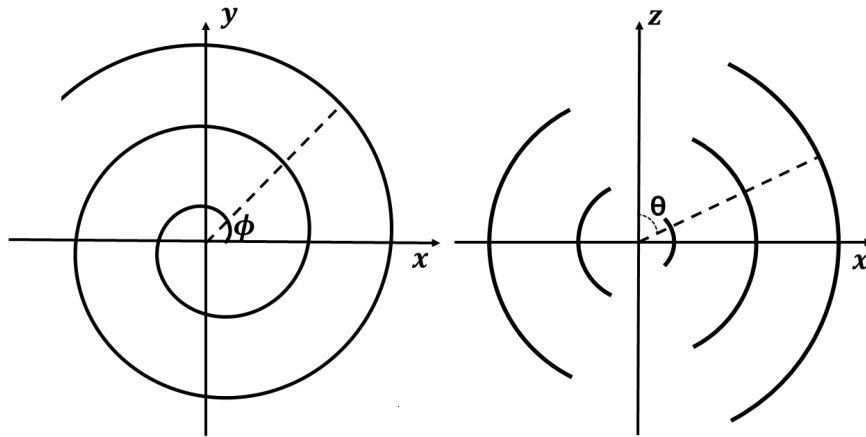
d) (4 points) Suppose a star passes exactly through the zenith; find the observer's latitudes  $\phi_{\max}$  and  $\phi_{\min}$  so that the sizes of the blind spots are maximal and minimal, respectively.

### T08. Evolved Star Spirals (25 points)

The red giant AFGL 3068 is located about  $1.0 \text{ kpc}$  from Earth. Its images reveal a nearly Archimedean spiral with a constant arm spacing, described by  $r = a + b\phi$  where  $a$  and  $b$  are constants, and  $\phi$  is azimuthal angle in radian.



Such a spiral structure can be formed around a binary system. A red giant with a fast stellar wind orbiting the binary center of mass produces a thin, enhanced-density shell that expands radially at the wind velocity of  $V_w$  in the center of mass coordinate system. Its shapes on both the orbital plane ( $Oxy$ ) and a perpendicular plane ( $Oxz$ ) are shown below, where  $\phi$  and  $\theta$  denote the azimuthal and polar angles, respectively. Let  $r_p$  and  $V_p$  be the orbital radius and velocity of the red giant, respectively.



**Part 1 (10 points):**

- (2 points) The on-sky arm spacing of AFGL 3068 is 2.29 arcsec. Determine the constant  $b$  in  $\text{au rad}^{-1}$ .
- (4 points) At every point on the red giant trajectory, we consider only the wind moving radially outward with respect to the center of the binary system. Show that, on the orbital plane, the shell forms an Archimedean spiral structure with  $a = r_p$  and  $b = \frac{r_p V_w}{V_p}$ .
- (2 points) The wind velocity of AFGL 3068 is  $V_w = 14.0 \text{ km s}^{-1}$ . Calculate its orbital period  $T$  in years.

d) (2 points) The total mass of the binary system is  $6.6 M_{\odot}$ . Calculate the separation  $d$  between the two stars in au.

**Part 2 (15 points):** Assume that the orbital plane is inclined by an angle  $i$  with respect to the sky plane. The  $x$ -axis is along the intersection of the orbital and sky planes.  $r_p$  is negligible compared with the spiral size.

a) (6 points) Express the coordinates  $(x, y, z)$  and velocities  $(V_x, V_y, V_z)$  of a shell element in terms of the parameters  $(\phi, \theta)$ .

b) (7 points) Show that all shell elements with line-of-sight velocity  $V_{\text{los}} = 0$  satisfy the equation of an ellipse on the sky plane with two semi-axes of  $r \sin \theta$  and  $r \sin \theta / \cos i$ .

c) (2 points) Recent CO line images at  $V_{\text{los}} = 0$  channel show that the spiral pattern of AFGL 3068 is slightly elongated relative to an Archimedean spiral. The elongated pattern is well described by ellipses with a ratio of 1.1 between the two semi-axes. Determine the inclination angle  $i$ .

#### T09. Gravitational Lensing and Delay (20 points)

##### Part 1 (15 points): Einstein Ring

In February 2025, the Euclid mission reported an Einstein ring around the nearby elliptical galaxy NGC 6505, located at a distance  $d_L$  from Earth (Figure T09.1). The ring is formed when light from a source galaxy at a distance  $d_S \gg d_L$  passes near NGC 6505 and is bent by its gravitational field.

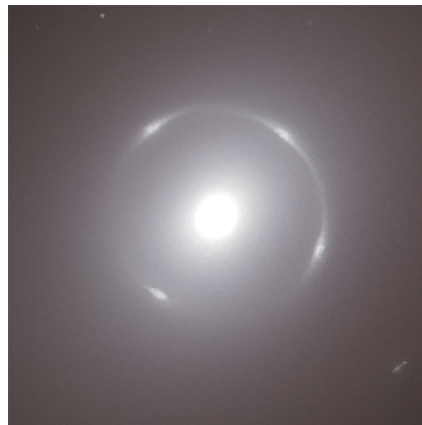


**Figure T09.1.** Euclid image of NGC 6505 and the Einstein ring surrounding it.  
Source: ESA/Euclid/Euclid Consortium/NASA.

a) (2 points) The measured redshifts of the source and lens galaxies are 0.406 and 0.042, respectively. Calculate  $d_S$  and  $d_L$  (in Mpc) assuming that the Newtonian approximation is valid and distance is proportional to redshift.

b) (3 points) In the linear approximation, as shown in Figure T09.2, the total deflection angle of a light ray emitted by the source and reaching the observer is given by  $2\alpha = 2R^*/b$ , where  $R^*$  is the Schwarzschild radius of the lens and  $b$  is the distance of closest approach of the light ray to the center of the lens. Express the angular radius  $\theta$  of the Einstein ring as a function of  $R^*$ ,  $d_S$  and  $d_L$ .

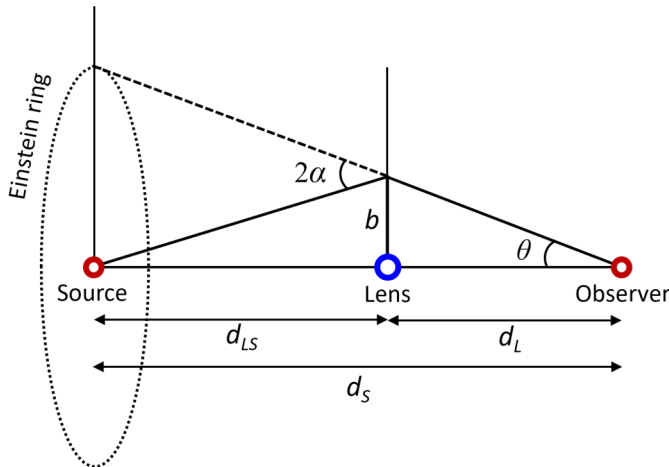


**Figure T09.2.** Simplified geometry of Einstein ring formation.

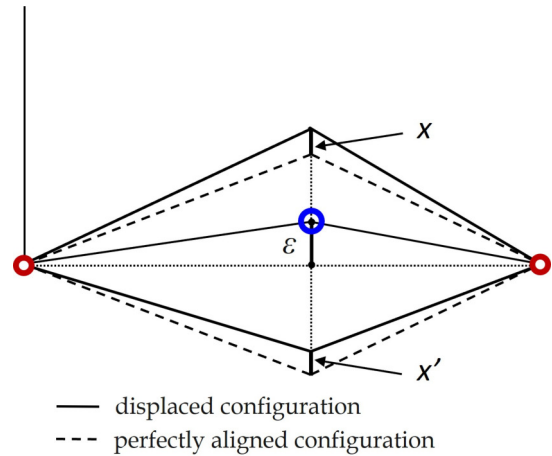


**Figure T09.3.** Geometry obtained when the lens galaxy is displaced.

c) (3 points) Euclid measures the angular radius of the ring around NGC 6505 to be 2.5 arcsec. Determine the distance of closest approach  $b$  (in parsecs) and the mass of the lens galaxy  $M$  (in units of  $M_\odot$ ).

d) (7 points) Suppose that the lens is displaced by a small distance  $\varepsilon$  from the line joining the source and the observer ( $\varepsilon \ll b$ ) as illustrated in Figure T09.3. Determine the quantities  $x$  and  $x'$  (indicated in the figure), as well as the angular displacement  $\Delta\theta$  of the Einstein ring image.

## Part 2 (5 points): Gravitational Delay

Let's now consider light being lensed by the Sun.

a) (2 points) Calculate the total deflection angle  $2\alpha$  (in arcsec) of a light ray grazing the Sun's surface.

b) (3 points) A light ray propagating in the gravitational field of a massive object experiences an additional travel time, known as the gravitational delay ( $\delta$ ).

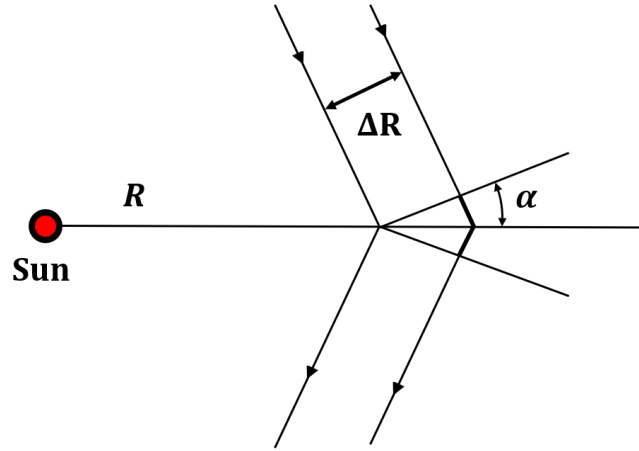


**Figure T09.4.** Simplified geometry of light bending and gravitational delay.

Consider two light rays passing near the Sun with distances of closest approach ( $R$ ) differing by a small amount  $\Delta R$  (Figure T09.4). The ray with the larger distance of closest approach travels an additional path length of approximately  $2\alpha\Delta R$ , resulting in a difference in gravitational delay  $\Delta\delta \sim 2\alpha\Delta R/c$ . Therefore, the absolute value of the derivative of the gravitational delay is

$$\frac{d\delta}{dR} = \frac{2\alpha}{c}$$

Determine the gravitational delay for a light signal from a distant source passing near the Sun's surface and reaching the Earth.

#### T10. The Chevallier-Polarski-Linder Parametrization for Dynamical Dark Energy (20 points)

Consider dynamical dark energy, which is parameterized by an equation of state,

$$w_{\text{DE}}(z) = \frac{P_{\text{DE}}}{c^2 \rho_{\text{DE}}}$$

where  $w_{\text{DE}}(z)$  is a function of redshift  $z$ , while  $P_{\text{DE}}$  and  $\rho_{\text{DE}}$  are the pressure and density of dark energy, respectively.

a) (5 points) In the spatially flat Friedmann-Lemaitre-Robertson-Walker spacetime, the equation of conservation of dark energy is

$$\dot{\rho}_{\text{DE}} + 3H \left( \rho_{\text{DE}} + \frac{P_{\text{DE}}}{c^2} \right) = 0$$

where  $H(t) = \frac{\dot{a}(t)}{a(t)}$  is the Hubble parameter,  $a(t)$  is the scale factor, and  $t$  is the cosmic time. Hereafter, an overdot denotes the time derivative of a quantity. Show that

$$\rho_{\text{DE}}(z) = \rho_{\text{DE},0} \exp \left[ \int_0^z \frac{3[1 + w_{\text{DE}}(z')]}{1 + z'} dz' \right]$$

where  $\rho_{\text{DE},0} = \rho_{\text{DE}}(t_0)$ , provided that  $\frac{a(t_0)}{a(t)} = 1 + z$ , with  $t_0$  being the present cosmic time. And  $a(t_0) = 1$  will be chosen for convenience.

b) (5 points) Consider the Friedmann equation,

$$3H^2 = 8\pi G (\rho_{\text{m}} + \rho_{\text{DE}})$$

where  $\rho_{\text{m}}$  is the density of matter (ordinary and dark matter). When the pressure of matter is negligible, the equation of conservation of matter is

$$\dot{\rho}_{\text{m}} + 3H\rho_{\text{m}} = 0$$

Show that the evolution parameter  $E(z) = \frac{H(z)}{H_0}$  obeys

$$E^2(z) = \Omega_{\text{m},0}(1+z)^3 + \Omega_{\text{DE},0} \exp \left\{ \int_0^z \frac{3[1 + w_{\text{DE}}(z')]}{1 + z'} dz' \right\}$$

where  $\Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{\rho_{\text{crit},0}}$ ,  $\Omega_{\text{DE},0} = \frac{\rho_{\text{DE},0}}{\rho_{\text{crit},0}}$ ,  $\rho_{\text{crit},0} = \frac{3H_0^2}{(8\pi G)}$ ,  $H_0 = H(t_0)$ , and  $\rho_{\text{m},0} = \rho_{\text{m}}(t_0)$ .

c) (5 points) Consider the Chevallier-Polarski-Linder parametrization,

$$w_{\text{DE}}(a) = w_0 + w_a(1 - a)$$

where  $w_0$  and  $w_a$  are coefficients. Notice that  $a(t_0) = 1$ . Find the corresponding  $E^2(z)$ .

d) (5 points) Matter clumps when it dominates over dark energy. Determine the necessary condition on  $w_0 + w_a$  such that matter can clump in early times at high redshifts  $z \gg 1$ .

### T11. Physics of Jupiter's Interior (50 points)

Jupiter is the biggest planet in our Solar System. Its mass and radius are  $M_J = 1.90 \times 10^{27}$  kg and  $R_J = 6.99 \times 10^7$  m, respectively. Herein, we investigate Jupiter's internal structure using a simple model. As shown in Figure T11.1, this gas giant is assumed to contain two distinct regions: a dense core and a fluid envelope. While the dense core is entirely made of rock, the fluid envelope is wholly composed of hydrogen. The mass densities of rock ( $\rho_Z$ ) and hydrogen ( $\rho_H$ ) are given by

$$\rho_Z = \text{const}, \quad \rho_H = \sqrt{\frac{P}{K}}$$

where  $P$  is the hydrostatic pressure and  $K$  is the polytropic coefficient. The space outside Jupiter is considered an absolute vacuum. The effects of self-rotation are neglected to achieve perfect spherical symmetry. These approximations enable us to study Jupiter's structural properties via a single spatial variable: the distance  $r$  from the center  $O$ .

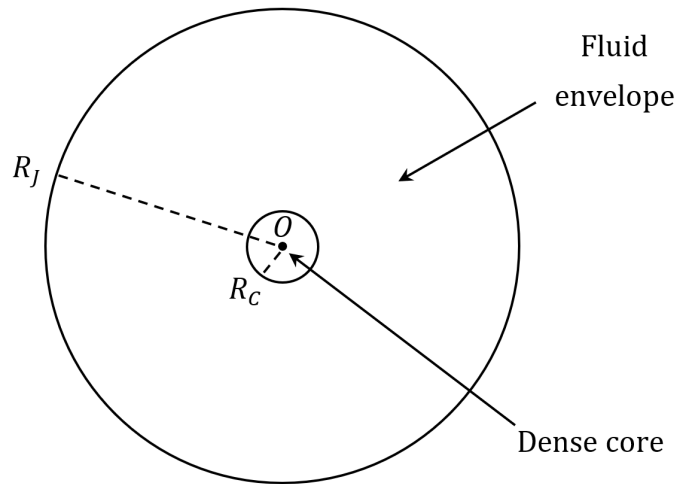


**Figure T11.1.** Preliminarily modeling Jupiter's interior.

**Part 1 (23 points):** Assume that the dense core is not eroded by the fluid envelope. In this case,  $P$  decreases from  $P_O = 7.13 \times 10^{12}$  Pa at the center ( $r = 0$ ) to  $P_C = 3.98 \times 10^{12}$  Pa at the core-envelope boundary ( $r = R_C = 1.05 \times 10^7$  m). The polytropic coefficient of pure hydrogen is  $K = 2.12 \times 10^5 \text{ m}^5 \text{ kg}^{-1} \text{ s}^{-2}$ .

- a) (6 points) Find the value of  $\rho_Z$ .
- b) (6 points) Since the fluid envelope is equivalent to a simple polytrope ( $P \propto \rho_H^2$ ), its density depends on the distance as

$$\rho_H = \frac{A}{r} \cos\left(\frac{r}{R} + \varphi\right)$$

where  $R$  is the characteristic radius,  $A$  is the amplitude factor, and  $\varphi$  is the phase constant. Given that

$$R = 2.25 \times 10^7 \text{ m}, \quad A > 0, \quad -\pi \leq \varphi \leq \pi$$

Determine  $A$  and  $\varphi$ .

- c) (8 points) Plot the density and pressure versus the distance for the whole of Jupiter's interior. Each graph must consist of at least six distinct points.
- d) (3 points) Calculate the mass of the dense core  $M_Z$  and the mass of the fluid envelope  $M_H$ .

**Part 2 (27 points):** Recent studies show that giant planets may undergo core erosion due to the high solubility of rock in hydrogen. For Jupiter, this process helps redistribute heavy and light materials throughout its interior. When the dense core disappears, Jupiter becomes an ideal mixture of rock and hydrogen at equilibrium. Neglect all possible chemical reactions. For a given position inside Jupiter, the mass fraction of rock  $Z$  is defined by

$$Z = \frac{dm_Z}{dm_Z + dm_H}$$

where  $dm_Z$  and  $dm_H$  are the differential masses of rock and hydrogen, respectively.

a) (6 points) Let  $\rho$  be the mass density of the mixture at the distance  $r$ . Prove that  $\rho$  is related to  $P$  via

$$\rho = \frac{\rho_Z \sqrt{P}}{Z \sqrt{P} + (1 - Z) \rho_Z \sqrt{K}}$$

b) (7 points) Figure T11.2 shows how  $Z$  depends on  $r$ . Based on the  $Z - r$  relation, we can divide Jupiter's interior into three different regions: the dilute core ( $Z = Z_1 = 7.28 \times 10^{-2}$ ,  $0 \leq r \leq R_1$ ), the transition layer ( $Z$  decreases linearly with increasing  $r$ ,  $R_1 \leq r \leq R_2$ ), and the hydrogen-dominated envelope ( $Z = Z_2 = 1.53 \times 10^{-2}$ ,  $R_2 \leq r \leq R_J$ ). In the transition layer,  $\rho$  is well approximated as a linear function of  $r$ , and  $P$  drops from  $P_1 = 2.25 \times 10^{12}$  Pa at  $R_1 = 2.87 \times 10^7$  m to  $P_2 = 8.43 \times 10^{11}$  Pa at  $R_2 = 4.40 \times 10^7$  m. The polytropic coefficient of hydrogen in the mixture is  $K = 2.20 \times 10^5 \text{ m}^5 \text{ kg}^{-1} \text{ s}^{-2}$ . Determine the mass of the transition layer  $M_{12}$ .

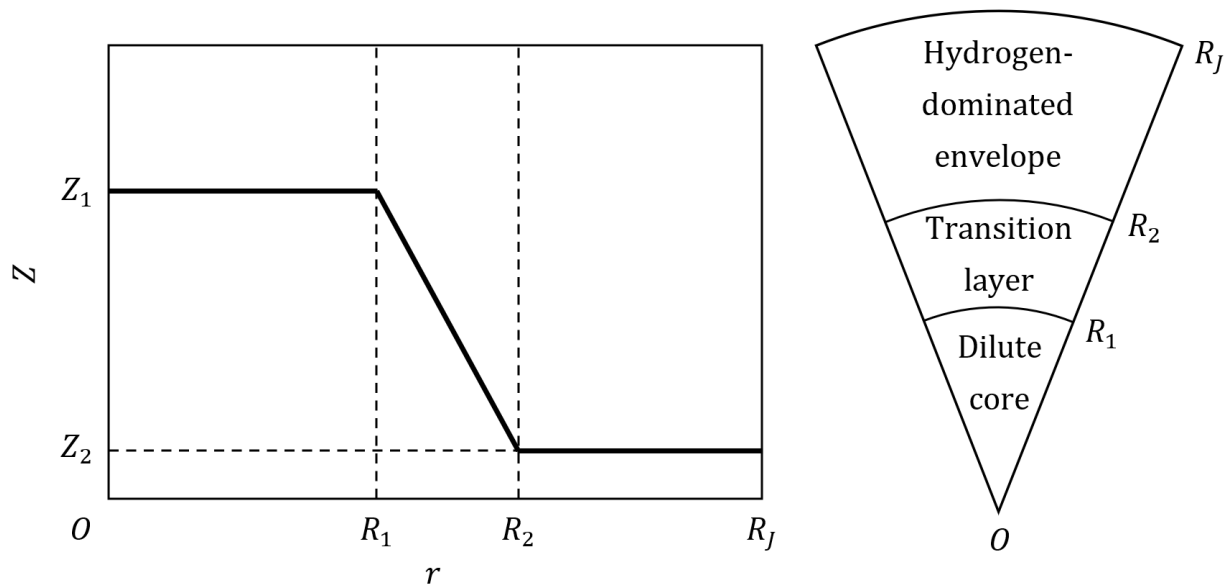


**Figure T11.2.** (Left) The mass fraction of rock versus the distance from the center. (Right) The cross-sectional view of Jupiter after core erosion.

c) (14 points) Calculate the mass of the dilute core  $M_1$  and the mass of the hydrogen-dominated envelope  $M_2$ .

### T12. Superluminous Galaxies in the Early Universe (50 points)

JWST has identified exceptionally compact, luminous galaxies at redshifts  $z > 7$ . To understand the nature of these sources, we must analyze the observational data and test whether they are consistent with basic astrophysical constraints. The table below lists the constants and parameters used in this problem.

|                                                           |                                                              |
|-----------------------------------------------------------|--------------------------------------------------------------|
| Star formation rate (SFR) to Luminosity conversion factor | $k = 1.0 \times 10^{10} L_{\odot} M_{\odot}^{-1} \text{ yr}$ |
| Effective opacity of free electrons                       | $\kappa_e = 0.04 \text{ m}^2 \text{ kg}^{-1}$                |
| Effective opacity of the dusty interstellar medium        | $\kappa_{\text{dust}} = 1.5 \text{ m}^2 \text{ kg}^{-1}$     |
| Optical-to-bolometric flux correction factor              | $BC = F_{\text{bol}}/F_{\text{opt}} = 9.26$                  |

#### Part 1 (13 points): Cosmological Surface Brightness Dimming

Let  $L$  be the intrinsic bolometric luminosity and  $R$  be the intrinsic physical radius of a galaxy. The intrinsic surface luminosity density is defined as

$$\Sigma_{\text{int}} = \frac{L}{\pi R^2}$$

A telescope observes this galaxy subtending with a solid angle  $\Omega_{\text{obs}}$  and measures a flux  $F_{\text{obs}}$ . The observed surface brightness is

$$\Sigma_{\text{obs}} = \frac{F_{\text{obs}}}{\Omega_{\text{obs}}}$$

In cosmology, the luminosity distance ( $D_L$ ) is defined via

$$F_{\text{obs}} = \frac{L}{4\pi D_L^2}$$

The angular diameter distance ( $D_A$ ) is defined so that the observed angular radius is

$$\theta = \frac{R}{D_A}$$

- a) (10 points) Derive the relation between  $D_L$ ,  $D_A$  and redshift  $z$ .
- b) (3 points) Derive an expression for  $\Sigma_{\text{obs}}$  as a function of  $\Sigma_{\text{int}}$  and redshift  $z$ .

### Part 2 (8 points): Dissecting JWST Data

JWST observed an ultraluminous galaxy at a redshift of  $z = 7.26$ .

- The observed rest-frame optical flux at Earth (attenuated by cosmic dust) is  $F_{\text{obs,opt}} = 5.72 \times 10^{-19} \text{ W m}^{-2}$ .
- The dust extinction derived from the spectrum is  $A_V = 2.79 \text{ mag}$ .
- The measured angular radius of the source on the image is  $\theta = 0.05 \text{ arcsec}$ .

Calculate the intrinsic bolometric surface luminosity density of this galaxy,  $\Sigma_{\text{int}}$ , in units of  $L_{\odot} \text{ kpc}^{-2}$ .

### Part 3 (18 points): Eddington Dynamical Model for the Star-Forming Disk

Assume the entire intrinsic radiative energy of this galaxy originates from extreme star formation in an optically thick disk of cold gas and dust. Radiation from young stars generates tremendous pressure that drives cosmic dust particles outward, perpendicular to the disk midplane. Due to the extreme density, gas and dust are perfectly dynamically coupled via collisions. The radiation pressure acting on the dust effectively drags the entire gas mass along as a single unified fluid. Simultaneously, gravity pulls this fluid back toward the midplane. Assume the gravitational field of the disk is governed by the total dynamical surface mass density, given as  $\Sigma_{\text{tot}} = 2.3 \times 10^{10} M_{\odot} \text{ kpc}^{-2}$ .

- a) (6 points)  $\Sigma_{\text{int\_max}}$  denotes the disk's Eddington surface luminosity density. Prove the acceleration driven by radiation pressure is  $g_{\text{rad}} = \frac{\kappa_{\text{dust}} \Sigma_{\text{int\_max}}}{c}$  and the acceleration governed by the disk's gravity is  $g_z = 2\pi G \Sigma_{\text{tot}}$ .
- b) (7 points) Calculate the SFR surface density at Eddington limit ( $\Sigma_{\text{Edd}}$ ) of this disk (in units of  $M_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$ ) such that it remains gravitationally bound against a radiation-driven blowout.
- c) (5 points) Calculate the ratio  $\Sigma_{\text{SFR\_actual}} / \Sigma_{\text{Edd}}$ , where  $\Sigma_{\text{SFR\_actual}}$  is the actual SFR surface density of this galaxy. If the observed luminosity is purely stellar in origin, is this system dynamically stable?

### Part 4 (11 points): Clues from High-Resolution Spectroscopy

Astronomers analyzed the NIRSpect spectrum and discovered that the emission predominantly originates from a hot, highly ionized plasma, where continuous recombination processes emit Hydrogen lines exhibiting anomalous broadening with a velocity dispersion of  $v = 3400 \text{ km s}^{-1}$ . Furthermore, the radius of this emission region is indirectly estimated to be merely  $R_{\text{em}} = 0.18 \text{ pc}$ .

a) (4 points) Calculate the dynamical mass  $M$  confined within this region (in  $M_{\odot}$ ). Based on your result, choose the specific type of astrophysical object.

- A. A dwarf galaxy
- B. A massive star
- C. A super massive black hole
- D. A globular cluster

b) (7 points) Quantitatively evaluate the radiative stability of the system with respect to its observed luminosity. Is this system radiatively stable?