

T01. Interstellar Object (10 points)

An interstellar object (ISO) enters the solar system on a hyperbolic orbit. The distance from the Sun to the asymptote of the ISO's trajectory is $p = 1.00$ au, and the ISO has an asymptotic speed $V_\infty = 26 \text{ km s}^{-1}$ in the solar reference frame.

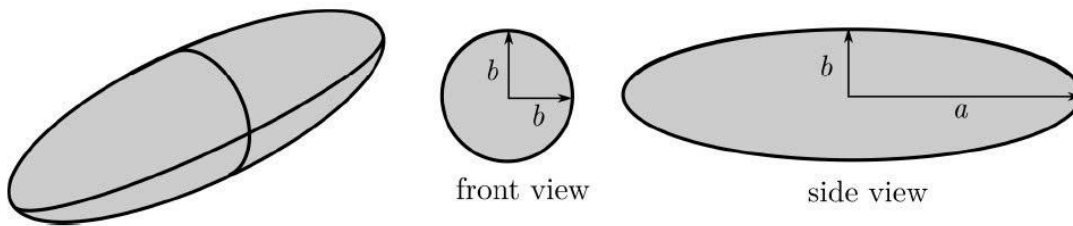
a) (5 points) Determine the perihelion distance r_p of the ISO (in meters).

Solution	Point
Based on the law of conservation of energy, we have	
$\frac{mV_\infty^2}{2} = \frac{mv_p^2}{2} - \frac{GmM_\odot}{r_p} \quad (1)$	1
where m is the mass of the ISO, r_p is the perihelion distance, and v_p is the perihelion velocity. Besides, using the law of conservation of angular momentum gives us	
$mpV_\infty = mr_p v_p \quad (2)$	1
Combining Eq. (1) and Eq. (2) leads to	
$\frac{p^2 V_\infty^2}{r_p^2} - \frac{2GM_\odot}{r_p} - V_\infty^2 = 0 \quad (3)$	1
Solving Eq. (3) with $r_p > 0$ provides	
$r_p = \frac{p^2 V_\infty^2}{GM_\odot + \sqrt{G^2 M_\odot^2 + p^2 V_\infty^4}} \quad (4)$	1
Our numerical result is	
$r_p = 5.05 \times 10^{10} \text{ m} \quad (5)$	1

b) (2 points) Calculate the equilibrium surface temperature T of the object at perihelion, assuming that the ISO is spherical, fast rotating, and has an ideal blackbody surface composed of water ice. In space the water ice sublimates at 120 K. Do we expect the water ice on ISO's surface to sublimate? Write YES / NO as the final answer.

Solution	Point
Let R_{ISO} be the radius of the ISO. Assuming an ideal blackbody and fast rotation, we have $\left(\frac{L_{\odot}}{4\pi r_p^2}\right) \pi R_{\text{ISO}}^2 = 4\pi R_{\text{ISO}}^2 \sigma T^4 \quad (6)$ Thus, the equilibrium surface temperature of the ISO is $T = \left(\frac{L_{\odot}}{16\pi\sigma r_p^2}\right)^{\frac{1}{4}} = 4.8 \times 10^2 \text{ K} \quad (7)$ This value is well above the sublimation temperature of water ice in vacuum, so sublimation is expected if exposed water ice is present.	1 0.5 0.5

c) (3 points) Observations show that the object's apparent magnitude varies periodically with a light-curve peak-to-peak amplitude of 2.0 mag due to its spinning, revealing the non-spherical shape of the ISO. Assuming that the object is an ellipsoid and the flux is proportional to the projected area during rotation and the ISO is rotating about its minor axis, which is perpendicular to the line-of-sight. Estimate the ratio of the major axis to minor axis of the object, a/b .



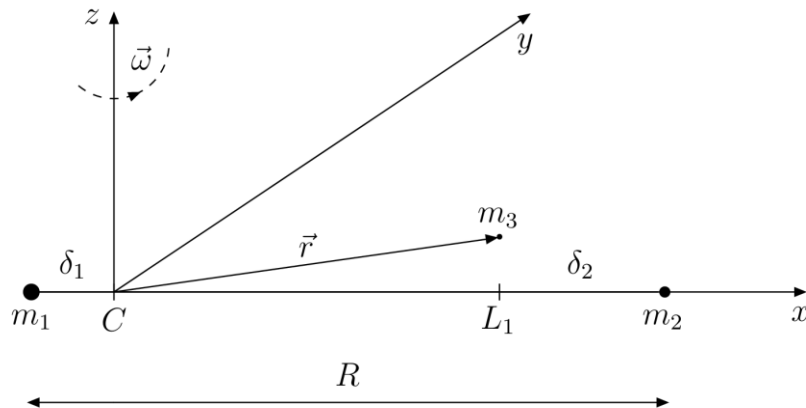
Solution	Point
The magnitude difference Δm is related to the flux ratio $F_{\text{max}}/F_{\text{min}}$ by $\Delta m = 2.5 \log_{10} \left(\frac{F_{\text{max}}}{F_{\text{min}}} \right) \quad (8)$ Since the brightness change is entirely due to projected area, we obtain $\frac{F_{\text{max}}}{F_{\text{min}}} = \frac{a}{b} \quad (9)$	1 1

where a/b is the major-to-minor axis ratio. Combining Eqs. (8) and (9) yields

$$\frac{a}{b} = 10^{\Delta m/2.5} = 6.3 \quad (10) \quad 1$$

T02. Oscillating Around a Lagrange Point (10 points)

A binary system, whose masses m_1 and m_2 are hierarchical, $m_1 \gg m_2$, and separated by a distance R , circularly orbits its center of mass, C . The rotating reference frame $Cxyz$ has the x -axis pointing from m_1 to m_2 and the z -axis oriented with the angular velocity of the system, $\vec{\omega}$. A satellite whose mass m_3 is much smaller than m_1, m_2 oscillates in the z -direction about the Lagrange point L_1 lying on the line segment that connects m_1 and m_2 . Let δ_1 and δ_2 be the distances from C to m_1 and L_1 to m_2 , respectively.



a) (5 points) Determine δ_1 and δ_2 as functions of m_1, m_2 , and R at the leading order.

Solution	Point
The center of mass obeys $\delta_1 m_1 = (R - \delta_1) m_2$ We rewrite $\delta_1 = \frac{m_2}{m_1} \left(1 + \frac{m_2}{m_1} \right)^{-1} R$	1

Using $(1+x)^\alpha \simeq 1+\alpha x$ for $x = \frac{m_2}{m_1} \ll 1$, we have $\frac{m_2}{m_1} \left(1 + \frac{m_2}{m_1}\right)^{-1} \simeq \frac{m_2}{m_1} \left(1 - \frac{m_2}{m_1}\right) \simeq \frac{m_2}{m_1}$ at the first order. Hence, $\delta_1 \simeq \frac{m_2}{m_1} R$	1
In the rotating frame, the net force at L_1 includes the gravitational forces, $-Gm_3m_1/(R-\delta_2)^2$ due to m_1 and Gm_3m_2/δ_2^2 due to m_2, and the centrifugal force $m_3\omega^2(R-\delta_1-\delta_2)$ with $\omega^2 = G(m_1+m_2)/R^3$ by the Kepler's law, where $CL_1 = R - \delta_1 - \delta_2$. The force balance implies $-\frac{Gm_1}{(R-\delta_2)^2} + \frac{Gm_2}{\delta_2^2} + \omega^2(R-\delta_1-\delta_2) = 0$ With the aid of $(1+x)^\alpha \simeq 1+\alpha x$ and $(1+x_1)(1+x_2) \simeq 1+x_1+x_2$ for $x , x_1 , x_2 \ll 1$, we approximate $\frac{1}{(R-\delta_2)^2} = \frac{1}{R^2} \left(1 - \frac{\delta_2}{R}\right)^{-2} \simeq \frac{1}{R^2} \left(1 + \frac{2\delta_2}{R}\right)$ and $\omega^2(R-\delta_1-\delta_2) = \frac{Gm_1}{R^2} \left(1 + \frac{m_2}{m_1}\right) \left(1 - \frac{\delta_1}{R} - \frac{\delta_2}{R}\right) \simeq \frac{Gm_1}{R^2} \left(1 - \frac{\delta_2}{R}\right)$ which uses the previous result $\frac{m_2}{m_1} \simeq \frac{\delta_1}{R}$. Substituting these approximations into the force balance equation gives $\delta_2 \simeq \left(\frac{m_2}{3m_1}\right)^{\frac{1}{3}} R$ <u>Note:</u> δ_1 is very smaller than δ_2 even though both are very smaller than R. Hence, δ_1 may be neglected when evaluating δ_2. Additionally, the Lagrange point L_1 may exist even if the mass hierarchy is large.	1.5
	1.5

b) (3 points) Find the equation of motion of the satellite along the z-axis.

Solution	Point
The satellite moving in the z-direction is not acted by the Coriolis and centrifugal forces, since $\vec{F}_{\text{Coriolis}} = -2m_3(\vec{\omega} \times \vec{v}) = 0 \text{ and } (\vec{F}_{\text{cent}})_z = -m_3[\vec{\omega} \times (\vec{\omega} \times \vec{r})]_z = 0$ Hence, the satellite experiences only the gravitational forces due to m_1 and m_2, projected on the z-axis as $-Gm_3m_1 \frac{z}{\rho_1^3} \text{ and } -Gm_3m_2 \frac{z}{\rho_2^3}$ where z is the z-coordinate of the satellite, and the distances between the satellite to two stars are $\rho_1 = \sqrt{(R - \delta_2)^2 + z^2}$ and $\rho_2 = \sqrt{\delta_2^2 + z^2}$. We obtain the equation of motion $\frac{d^2z}{dt^2} = -Gm_1 \frac{z}{\rho_1^3} - Gm_2 \frac{z}{\rho_2^3}$ where t is the time. The net force acts as a spring force which always pulls the satellite toward L_1, implying that the oscillation is stable. <i>Note: If the student does not discuss the Coriolis and Centrifugal forces but giving the correct net force and/or the correct equation of motion, the marks are accordingly added up.</i>	1 1 1

c) (2 points) Estimate the period T of small oscillations of the satellite in the z -direction for the Earth-Sun system.

Solution	Point
The small oscillations of satellite along z-direction imply $\rho_1 \simeq R - \delta_2$ and $\rho_2 \simeq \delta_2$, since $z \ll \delta_2$. Hence, the oscillation equation is periodic, $\frac{d^2z}{dt^2} + \omega_z^2 z = 0$ with angular frequency,	

$$\omega_z = \sqrt{\frac{Gm_1}{(R - \delta_2)^3} + \frac{Gm_2}{\delta_2^3}} \simeq 2 \sqrt{\frac{Gm_1}{R^3}} \simeq 2\omega$$

1

where we use the approximation $\delta_2 \simeq \left(\frac{m_2}{3m_1}\right)^{\frac{1}{3}} R \ll R$, as obtained.

We estimate the period of satellite for the Earth-Sun system,

$$T = \frac{2\pi}{\omega_z} \simeq \frac{1}{2} \frac{2\pi}{\omega} = \frac{1}{2} T_{\text{Earth}}$$

Since the period of the Earth about the Sun is $T_{\text{Earth}} = 1$ year, we get

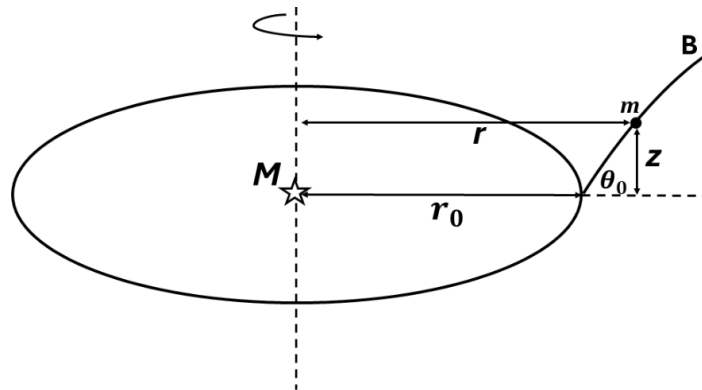
$$T \simeq \frac{1}{2} \text{year} = 6 \text{ months}$$

1

Note: Alternatively, the student can directly substitute for the Earth-Sun system, using the values $m_2 = m_{\text{Earth}} = 5.972 \times 10^{24}$ kg, $m_1 = m_{\text{Sun}} = 1.988 \times 10^{30}$ kg, $R = 1 \text{ au} = 1.496 \times 10^{11}$ m, $\delta_2 = 1.496 \times 10^9$ m, which is beyond the Earth radius, and $G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$. The period of satellite is

$$T = \frac{2\pi}{\sqrt{\frac{Gm_1}{(R - \delta_2)^3} + \frac{Gm_2}{\delta_2^3}}} \simeq \frac{\pi}{\sqrt{\frac{Gm_1}{R^3}}} \simeq 6 \text{ months}$$

T03. Stellar Winds Launched Along Magnetic Field Lines (10 points)



Angular momentum from a circular Keplerian accretion disk around a young star of mass M is removed by stellar winds. Wind particles slide frictionlessly along

magnetic field lines corotating with the disk. Consider a wind particle of mass m at radius r and height z moving along a magnetic field line that corotates with the disk at radius r_0 and is inclined at an angle θ_0 .

a) (2 points) In the frame corotating with the particle, derive $\Phi_{\text{eff}}(r, z)$, defined as the potential energy from the star's gravity and centrifugal force.

Solution	Point
At radius r_0 , the angular velocity is $\Omega_0 = \sqrt{\frac{GM}{r_0^3}}$	0.5
The gravitational potential energy of the particle is $\Phi_{\text{grav}}(r, z) = -\frac{GMm}{\sqrt{r^2 + z^2}}$	0.5
The centrifugal potential energy of the particle is $\Phi_{\text{cent}}(r) = -\frac{1}{2}m\Omega_0^2 r^2$	0.5
The effective potential energy of the particle is $\Phi_{\text{eff}}(r, z) = \Phi_{\text{grav}}(r, z) + \Phi_{\text{cent}}(r) = -\frac{GMm}{\sqrt{r^2 + z^2}} - \frac{1}{2}m\Omega_0^2 r^2$	0.5

b) (5 points) Introducing the variable $x = \frac{r-r_0}{r_0}$, write Φ_{eff} in the vicinity of r_0 to second-order approximation in x . Find the inclination angle θ_0 so that the particle can be pulled away from the disk. *Hint:* $(1 + u)^{-1/2} \approx 1 - \frac{u}{2} + \frac{3}{8}u^2$ with $|u| \ll 1$.

Solution	Point
Using the variable x , the coordinates of the particle in the vicinity of r_0 can be written as $r = r_0(1 + x), \quad z \approx r_0 x \tan \theta_0$	0.5
The gravitational potential energy expressed in terms of the variable x is	

$\Phi_{\text{grav}}(x) = -\frac{GMm}{r_0} [1 + 2x + x^2(1 + \tan^2 \theta_0)]^{-\frac{1}{2}}$	0.5
Defining $u = 2x + x^2(1 + \tan^2 \theta_0)$ and using the Taylor expansion $(1 + u)^{-1/2} \approx 1 - \frac{u}{2} + \frac{3}{8}u^2$ while retaining only terms up to second order in x, yields $\Phi_{\text{grav}}(x) \approx -m\Omega_0^2 r_0^2 \left[1 - \frac{1}{2}(2x + x^2 + x^2 \tan^2 \theta_0) + \frac{3}{8}(4x^2) \right]$ $\approx -m\Omega_0^2 r_0^2 \left[1 - x + x^2 \left(1 - \frac{\tan^2 \theta_0}{2} \right) \right]$	1.5
The centrifugal potential energy expressed in terms of the variable x is $\Phi_{\text{cent}}(x) = -\frac{1}{2} m\Omega_0^2 r_0^2 (1 + x)^2 = -m\Omega_0^2 r_0^2 \left(\frac{1}{2} + x + \frac{1}{2}x^2 \right)$	0.5
The effective potential energy expressed in terms of the variable x is $\Phi_{\text{eff}}(x) = -m\Omega_0^2 r_0^2 \left[\frac{3}{2} + x^2 \left(\frac{3}{2} - \frac{\tan^2 \theta_0}{2} \right) \right]$	1
The effective potential energy has a parabolic form and reaches an extremum at $x = 0$. For the particle to move away from the disk along the magnetic field line, this extremum must be a maximum of effective potential energy. This condition leads to $3 - \tan^2 \theta_0 > 0 \Rightarrow \theta_0 < 60^\circ$ <i>Note: Students can find $\theta_0 < 60^\circ$ from $F_{\text{cent}} > F_{\text{grav}}$, where F_{cent} and F_{grav} are centrifugal and gravitational forces, respectively.</i>	1

c) (3 points) At radius $r = r_A$, particles begin to decouple from magnetic field. The wind launched from r_0 carries mass-loss rate ΔM_w while the accretion rate is ΔM_{acc} . If all accretion angular momentum is removed by the wind, show that

$$\frac{\Delta M_{\text{acc}}}{\Delta M_w} = \left(\frac{r_A}{r_0} \right)^2$$

Solution	Point
At radius $r = r_A$, the particles still rotate at angular velocity Ω_0 . The rate of angular momentum removal by the stellar wind is $\Delta L_w = \Delta M_w \Omega_0 r_A^2$	1
The angular momentum releasing rate by accretion is $\Delta L_{acc} = \Delta M_{acc} \Omega_0 r_0^2$	1
Consider all the angular momentum released by accretion is removed through the stellar wind, we have $\Delta L_w = \Delta L_{acc} \Rightarrow \Delta M_w \Omega_0 r_A^2 = \Delta M_{acc} \Omega_0 r_0^2 \Rightarrow \frac{\Delta M_{acc}}{\Delta M_w} = \frac{r_A^2}{r_0^2}$	1

T04. Great Dimming of Betelgeuse (10 points)

Between late 2019 and early 2020, the red supergiant Betelgeuse experienced an unusual visual fading event known as the “Great Dimming” (Figure T04.1). One explanation is that the star ejected material that formed a dust clump, obscuring the stellar disk. Assume the dimming occurred at a normal pulsation minimum and was caused entirely by the cylindrical dust clump covering 80% of the stellar disk.

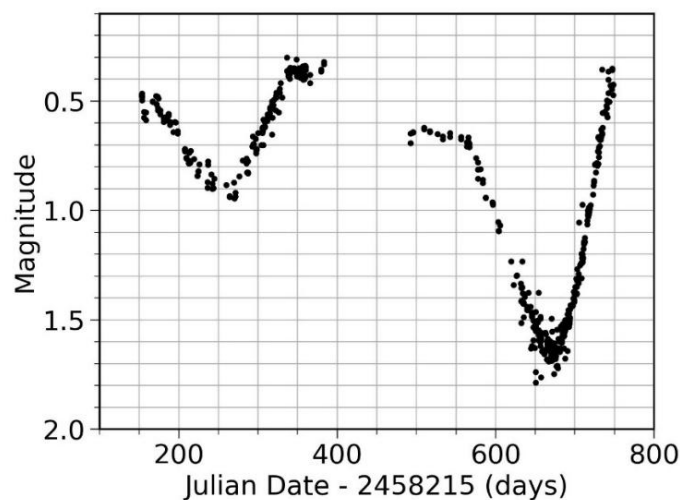


Figure T04.1. Visual light curve of Betelgeuse.

a) (2.5 points) Estimate the ratio (k_1) of the observed brightness at the minimum light to that at the normal pulsation minimum.

Solution			Point
From the graph, the magnitudes at minimum light of the pulsation cycle before and during Great Dimming are $m_1 = 0.9, \quad m_2 = 1.65$ The difference in magnitude between the two minimum light phases is $\Delta m = 0.75$			1
Half mark	Full mark	Half mark	
$0.65 \leq \Delta m < 0.70$	$0.70 \leq \Delta m \leq 0.80$	$0.80 < \Delta m \leq 0.85$	
Call F_0 the initial brightness of the star. The fractional decrease in the observed brightness caused by the dust clump is $k_1 = \frac{F}{F_0} = 10^{-\frac{\Delta m}{2.5}}$ Therefore, $k_1 = 0.50$			1
			0.5

b) (2.5 points) Estimate the fraction (k_2) of light transmitted through the obscured region.

Solution		Point
Call F_{clump} the brightness transmitted through the dust clump. The observed brightness of the star during the dimming is $F = 0.20F_0 + 0.80F_{\text{clump}}$ Hence, $k_2 = \frac{F_{\text{clump}}}{F_0} = \frac{\frac{F}{F_0} - 0.20}{0.80} = 0.38$		1.5
		1

c) (1.5 points) Calculate the optical depth τ of the dust clump.

Solution	Point
The brightness transmitted through the dust clump is $F_{\text{clump}} = F_0 e^{-\tau}$	1
Hence, the optical depth is $\tau = -\ln\left(\frac{F_{\text{clump}}}{F_0}\right) = 0.97$	0.5

d) (3.5 points) Assuming a mass absorption coefficient of the dust $\kappa = 1000. \text{ m}^2 \text{ kg}^{-1}$, a gas-to-dust mass ratio of 200., and a radius of Betelgeuse of $1000 R_{\odot}$, estimate the total mass (M_{eject}) of the ejected material in units of $M_{\odot}$.

Solution	Point
Call ρ the density of the dust clump, Δz the average extent of the clump along the line of sight. The optical depth is $\tau = \rho \kappa \Delta z$	1
The dust clump has a volume $V = 0.80 \pi R^2 \Delta z$	0.5
and a mass $M_{\text{dust}} = \rho V$	
Therefore, $M_{\text{dust}} = 0.80 \pi R^2 \frac{\tau}{\kappa}$	1
$M_{\text{dust}} = 5.9 \times 10^{-10} M_{\odot}$	0.5
The total mass of the ejected material is $M_{\text{eject}} = 201 M_{\text{dust}} = 1.2 \times 10^{-7} M_{\odot}$	0.5

T05. Primordial Black Holes (10 points)

A primordial black hole (PBH) is a hypothetical black hole, which is expected to be formed very soon after the Big Bang.

a) (5 points) Find the expression of the initial mass M of a PBH, provided that the cosmological density at a time t after the Big Bang is $\rho_c \sim 1/(Gt^2)$ and the PBH is assumed to be a homogeneous sphere having a radius identical to the Schwarzschild radius.

Solution	Point
Work out the mass density of the PBH, $\rho_{\text{PBH}} = \frac{M}{V_s} = \frac{M}{\frac{4}{3}\pi r_s^3}$ where the Schwarzschild radius is $r_s = \frac{2GM}{c^2}$ The above equation becomes $\rho_{\text{PBH}} = \frac{M}{\frac{4}{3}\pi \left(\frac{2GM}{c^2}\right)^3} = \frac{3c^6}{32\pi G^3 M^2}$ The PBHs can be formed if the cosmological density of the universe is of the same order as the density of the PBH, $\rho_c \sim \rho_{\text{PBH}}$ Then, we can figure out the desired relation, $\frac{1}{Gt^2} \sim \frac{3c^6}{32\pi G^3 M^2} \Rightarrow M \sim \frac{c^3 t}{6G}$ <u>Note:</u> If students use "=", "$\approx$", or "$\leq$" for the density relation above, the marks remain unchanged.	1 0.5 0.5 2 1

- Students can write the coefficient as $\sqrt{3/32\pi}$ or 1 instead of $1/6$.

b) (5 points) Estimate the initial mass and formation time of PBHs after the Big Bang such that they are still evaporating, provided that the black hole evaporation time is $\tau(M) = 10^{67} (M/M_{\odot})^3 (\text{yr})$.

Solution	Point
The PBHs still evaporating implies that the evaporation time must be greater than or equal to the Universe age, $\tau(M) \geq 13.8 (\text{Gyr})$ $\Rightarrow 10^{67} \left(\frac{M}{M_{\odot}} \right)^3 \geq 13.8 \times 10^9$ $\Rightarrow \left(\frac{M}{M_{\odot}} \right)^3 \geq 13.8 \times 10^{-58}$ $\Rightarrow M \geq 1.1 \times 10^{-19} \times M_{\odot}$	1.5 1
Then, using the result derived in part a), we can work out the formation time after the Big Bang of PBHs as follows $\frac{c^3 t}{6G} \gtrsim 1.1 \times 10^{-19} \times M_{\odot}$ $\Rightarrow t \gtrsim 6.6 \times 10^{-19} \times M_{\odot} \times \frac{G}{c^3}$ $\Rightarrow t \gtrsim 3.3 \times 10^{-24} (\text{s})$	1.5 1
<u>Note:</u> • Deduct 0.5 points if students use either “$\tau(M) = 13.8 \text{ Gyr}$” or “$\tau(M) > 13.8 \text{ Gyr}$”. • The above comparisons are in the same order of magnitude. Therefore, coefficients are not important or can be flexibly varied.	

T06. Solar Neutrinos: From Experimental Measurement to Core Temperature Modeling (20 points)

High-energy solar neutrinos from the pp-III chain are detected by Earth-based experiments like Super-Kamiokande to study the Sun's core properties.

a) (5 points) The pp-III neutrino flux on Earth is $\Phi = 5.1 \times 10^6 \text{ cm}^{-2}\text{s}^{-1}$. The Super-Kamiokande detector is a tank containing $5.0 \times 10^7 \text{ kg}$ of pure water (H_2O). The molar mass of H_2O is 18 g/mol . Its detection efficiency is $\eta = 44.5\%$. Assume that solar neutrinos only interact with the detector through elastic scattering with electrons. The average neutrino-electron interaction cross-section is $\sigma \approx 1.0 \times 10^{-44} \text{ cm}^2$ in water. Physically, σ can be understood as the effective target area a neutrino sees when approaching an electron; a larger σ means a higher probability of collision. Calculate the number of events captured in a single day N_v (24 hours).

Solution	Point
Number of moles of water: $n_{\text{H}_2\text{O}} = \frac{m}{M_{\text{H}_2\text{O}}} = \frac{5.0 \times 10^7}{18 \times 10^{-3}} \approx 2.8 \times 10^9 \text{ mol}$	0.5
Number of water molecules: $N_{\text{H}_2\text{O}} = n_{\text{H}_2\text{O}} \times N_A = (2.8 \times 10^9) \times (6.022 \times 10^{23})$ $\approx 1.7 \times 10^{33} \text{ molecules}$	0.5
Each water molecule contains 10 electrons, so the total number of electrons in the detector is: $N_e = N_{\text{H}_2\text{O}} \times 10 = (1.7 \times 10^{33}) \times 10 = 1.7 \times 10^{34} \text{ electrons}$	1
Expression to calculate the expected number of registered events in a time interval t: $N_v = \Phi N_e \sigma \eta t$	2
The expected number of events registered per day is:	

$N_\nu = (5.1 \times 10^6) \times (1.7 \times 10^{34}) \times (1.0 \times 10^{-44}) \times 0.445 \times 86400$ $\approx 33 \text{ events}$	1
--	---

b) (6 points) Neutrinos (ν) are detected indirectly via Cherenkov radiation emitted by scattered electrons (e^-), i.e., a product of the electron-neutrino scattering reaction ($\nu + e^- \rightarrow e^- + \nu$), in water (refractive index $n = 1.33$). This event occurs when the velocity of electron satisfies $v \geq c/n$. Hence, we must use relativistic mechanics. The relativistic kinetic energy of electron is given as $K = (\gamma - 1)m_e c^2$, where the Lorentz factor γ is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The post-scattering energy of the neutrino (E'_ν) is related to its pre-scattering counterpart (E_ν) as

$$E'_\nu = \frac{E_\nu}{1 + \frac{E_\nu}{m_e c^2} (1 - \cos\theta)}$$

where θ is the scattering angle of the neutrino. Assuming the electron's initial kinetic and binding energies are negligible, determine the minimum initial neutrino energy ($E_{\nu, \min}$) in MeV required to generate a Cherenkov radiation event.

Solution	Point
For the Cherenkov effect to occur, the electron velocity must satisfy the condition: $v \geq \frac{c}{n} \Rightarrow \frac{v}{c} \geq \frac{1}{n}$ At the threshold where radiation begins, we have $\gamma_{\min} = \frac{1}{\sqrt{1 - \frac{1}{n^2}}} = \frac{n}{\sqrt{n^2 - 1}}$	1

The minimum kinetic energy that an electron should have for the Cherenkov effect to occur is:

$$K_{\text{Cherenkov}} = \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) m_e c^2$$

Substituting the numerical values $n = 1.33$ and $m_e c^2 = 0.511 \text{ MeV}$:

$$K_{\text{Cherenkov}} = \left(\frac{1.33}{\sqrt{1.33^2 - 1}} - 1 \right) \times 0.511 = 0.264 \text{ MeV}$$

The kinetic energy received by the electron is:

$$K_e = E_\nu - E'_\nu$$

For K_e to reach its maximum, the neutrino energy after scattering E'_ν must hit its minimum value. E'_ν is minimized when $(1 - \cos\theta)$ reaches its maximum, meaning $\cos\theta = -1$ (backscattering at 180°). Thus:

$$K_{e,\text{max}} = E_\nu - \frac{E_\nu}{1 + \frac{2E_\nu}{m_e c^2}} = \frac{2E_\nu^2}{m_e c^2 + 2E_\nu}$$

E_ν is minimum once

$$K_{e,\text{max}} = K_{\text{Cherenkov}} = 0.264 \text{ MeV}$$

Solve the equation:

$$\frac{2E_\nu^2}{0.511 + 2E_\nu} = 0.264 \Rightarrow 2E_\nu^2 - 0.528E_\nu - 0.135 = 0$$

$$\Rightarrow E_{\nu,\text{min}} \approx 0.423 \text{ MeV or } E_{\nu,\text{min}} \approx -0.159 \text{ MeV}$$

Get the positive root because energy should be positive:

$$E_{\nu,\text{min}} \approx 0.423 \text{ MeV}$$

c) (9 points) The solar pp-III reaction rate $R(T)$ depends on the core temperature T via the Gamow equation $R(T) = R_0 e^{-BT^{-\frac{1}{3}}}$, where R_0 is a constant, $B = 17757 \text{ K}^{1/3}$ is the Gamow factor, and $T = 1.5 \times 10^7 \text{ K}$ is the steady-state core temperature.

Suppose the temperature fluctuates slightly ($T \rightarrow T + \Delta T$), causing a small fractional reaction rate change ($R \rightarrow R + \Delta R$). The relation between the fractional change in the reaction rate and temperature is in the form $\frac{\Delta R}{R} = \alpha \frac{\Delta T}{T}$, where α is the thermal sensitivity coefficient. Express α in terms of B and T . If the number of events captured by the Super-Kamiokande detector in a single day increases by 2%, calculate the corresponding percentage change in the Sun's core temperature.

Hints: $(1 + x)^\delta \approx 1 + \delta x$ and $\ln(1 + x) \approx x$ for $|x| \ll 1$.

Solution	Point
Express: $R + \Delta R = R(T + \Delta T) = R_0 e^{-B(T + \Delta T)^{-1/3}}$	1
Express the ratio: $1 + \frac{\Delta R}{R} = e^{-BT^{-1/3} \left[\left(1 + \frac{\Delta T}{T}\right)^{-1/3} - 1 \right]}$	1
Take the natural logarithm (ln) of both sides: $\ln \left(1 + \frac{\Delta R}{R} \right) = -BT^{-1/3} \left[\left(1 + \frac{\Delta T}{T} \right)^{-1/3} - 1 \right]$	1
The left-hand side is: $\ln \left(1 + \frac{\Delta R}{R} \right) \approx \frac{\Delta R}{R}$	0.5
The right-hand side is: $\left[\left(1 + \frac{\Delta T}{T} \right)^{-1/3} - 1 \right] \approx \left(1 - \frac{1}{3} \frac{\Delta T}{T} \right) - 1 = -\frac{1}{3} \frac{\Delta T}{T}$	0.5
Combine these two approximations: $\frac{\Delta R}{R} \approx -BT^{-1/3} \left(-\frac{1}{3} \frac{\Delta T}{T} \right)$ $\Rightarrow \frac{\Delta R}{R} \approx \left(\frac{1}{3} BT^{-1/3} \right) \frac{\Delta T}{T}$	

$\Rightarrow \alpha = \frac{1}{3}BT^{-1/3}$ <i>Note: If a student correctly determines α by taking the derivative of $R(T)$, give 5 points.</i> $\begin{aligned}\frac{dR}{dT} &= R_0 \exp(-BT^{-1/3}) \cdot (-B) \cdot \left(-\frac{1}{3}\right) T^{-4/3} \\ &= R(T) \cdot \frac{B}{3} T^{-4/3} \text{ (2 points)}\end{aligned}$ $\Delta R \approx \frac{dR}{dT} \Delta T = R \cdot \frac{B}{3} T^{-4/3} \Delta T \text{ (1 point)}$ $\frac{\Delta R}{R} \approx \frac{B}{3} T^{-4/3} \Delta T = \frac{B}{3T^{1/3}} \cdot \frac{\Delta T}{T} \text{ (1 point)}$ $\alpha = \frac{1}{3}BT^{-1/3} \text{ (1 point)}$	1
The fractional change in the reaction rate is: $\frac{\Delta R}{R} = \frac{\Delta \Phi}{\Phi} = \frac{\Delta N_v}{N_v} = 2\% = 0.02$	2
The thermal sensitivity coefficient is: $\alpha = \frac{1}{3}BT^{-\frac{1}{3}} = \frac{17757}{3 \times (1.5 \times 10^7)^{\frac{1}{3}}} \approx 24$	1
The corresponding percentage change in the Sun's core temperature is: $\frac{\Delta T}{T} = \frac{\Delta R}{R} \frac{1}{\alpha} = \frac{0.02}{24} \approx 0.083\%$ <i>Note: If a student bypasses the algebraic linearization proof and directly substitutes the numerical constants and $\frac{\Delta R}{R} = 0.02$ into the original unapproximated exponential ratio equation to solve for $\frac{\Delta T}{T}$ numerically, award 4 points provided they correctly calculate $\frac{\Delta T}{T} \approx 0.083\%$.</i>	1

T07. Zenith Blind Spot (15 points)

An observer in Hanoi (latitude of exactly $\phi = 21^\circ\text{N}$) uses an altazimuth telescope to observe Zeta Tauri (declination $\delta = +21^\circ 08' 33''$) with a maximum azimuthal tracking speed of $1.0^\circ/\text{s}$. As an object transits close to the zenith, the required tracking angular velocity increases sharply. If the required angular velocity exceeds the motor's maximum speed, the telescope loses its target. This problem is known as the "Zenith blind spot" or the "Gimbal lock". The position of the star is defined by its azimuth (A), zenith distance (z), and hour angle (H). Note that A is measured Eastward from North and H is measured Westward from South.

a) (4 points) Show that for the telescope to be able to track the star, the required azimuthal angular velocity at meridian transit is given by $\omega_A = -\omega_\oplus \frac{\cos \delta}{\sin z}$ where $\omega_\oplus = 15^\circ/\text{hr}$ is the Earth's sidereal rotation rate.

Solution	Point
Using the Spherical Law of Sines for the PZS (Pole-Zenith-Star) triangle: $\frac{\sin(90 - \delta)}{\sin A} = -\frac{\sin z}{\sin H} \Rightarrow \sin A \sin z = -\sin H \cos \delta$ (The "-" sign appears because H is measured Westward while A is measured Eastward) Taking the time derivative: $\frac{d}{dt}(\sin A \sin z) = \frac{d}{dt}(-\sin H \cos \delta)$ $\Rightarrow \cos A \sin z \frac{dA}{dt} + (\sin A \cos z) \frac{dz}{dt} = -(\cos H \cos \delta) \frac{dH}{dt}$	1
At transit: $H = 0 \Rightarrow \cos H = 1; A = 0 \Rightarrow \cos A = 1 \text{ and } \sin A = 0$	1
Since: $\frac{dH}{dt} = \omega_\oplus; \sin z \frac{dA}{dt} = -\omega_\oplus \cos \delta \Rightarrow \omega_A = \frac{dA}{dt} = -\omega_\oplus \frac{\cos \delta}{\sin z}$	1

b) (4 points) Calculate the minimum zenith distance ($z_{\min}$) reached by Zeta Tauri during its transit and the radius of the blind spot (z_{limit}) in arcseconds. Show that Zeta Tauri falls into the blind spot of the telescope during its transit over Hanoi.

Solution	Point
The minimum zenith distance: $z_{\min} = \phi - \delta = 21.00^\circ - 21^\circ 08' 33'' = 513''$	1
At the boundary of the blind zone, the azimuthal tracking speed of the telescope reaches its maximum $\omega_A = 1.0^\circ/\text{s}$, $\omega_\oplus = 15^\circ/\text{hr} = 0.00417^\circ/\text{s}$.	1
$\sin z_{\text{limit}} = \left \frac{\omega_\oplus \cos \delta}{\omega_A} \right = \left \frac{0.00417 \cos(21^\circ 08' 33'')}{1.0} \right \approx 0.00389$ $\Rightarrow z_{\text{limit}} \approx 802''$	1
Since $z_{\min} < z_{\text{limit}}$, Zeta Tauri will fall into the blind spot, and the telescope mount will fail to track it during transit.	1

c) (3 points) Assuming the star's path across the sky near the zenith can be approximated as a straight line, calculate the total time Δt (in seconds) that Zeta Tauri spends inside the telescope's blind spot.

Solution	Point
The blind spot forms a circle of radius z_{limit} centered on the zenith. The star's path is a chord across this circle, passing at a minimum distance of $z_{\min}$ from the center. Using the Pythagorean theorem for small angles: Total angular path: $L = 2\sqrt{z_{\text{limit}}^2 - z_{\min}^2} = 2\sqrt{802^2 - 513^2} = 1233''$	1
The star's apparent angular velocity is the Earth's rotation rate scaled by the cosine of the declination: $v_{\text{star}} = \omega_\oplus \cos \delta = 15''/\text{s} \times \cos(21^\circ 08' 33'') = 13.99''/\text{s}$	1
The time duration:	

$\Delta t = \frac{L}{v_{\text{star}}} = \frac{1233''}{13.99''/\text{s}} = 88 \text{ s}$	1
---	---

d) (4 points) Suppose a star passes exactly through the zenith; find the observer's latitudes ϕ_{max} and ϕ_{min} so that the sizes of the blind spots are maximal and minimal, respectively.

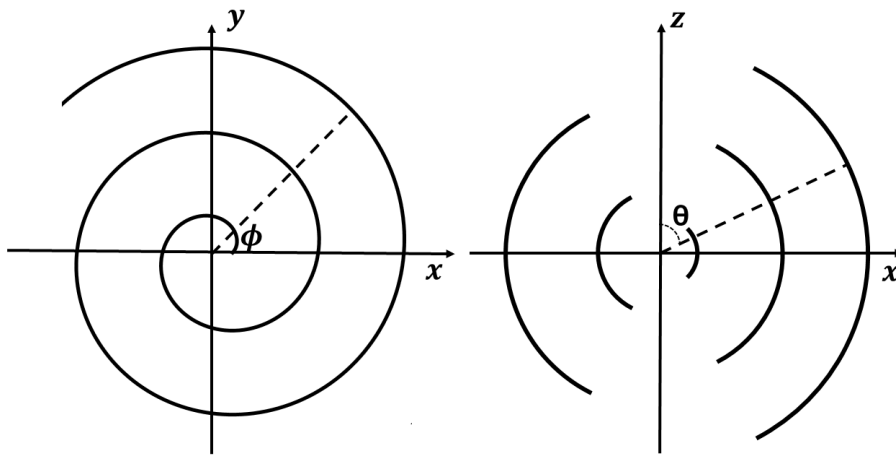
Solution	Point
From b), for a given maximum motor speed (ω_A): $\sin z_{\text{limit}} \propto \cos \delta$ For objects passing exactly through the zenith: $\delta = \phi$ Thus, the size of the blind spot scales with latitude: $\sin z_{\text{limit}} \propto \cos \phi$ At the Equator, ϕ_{max} is slightly above 0°, $\cos \phi_{\text{max}} \sim 1$, the blind spot reaches its maximum size. At the Poles, $\phi_{\text{min}} = \pm 90^\circ$, $\cos \phi_{\text{min}} = 0$, the blind spot completely disappears. <i>Physically, an altazimuth mount at the pole functions exactly like an equatorial mount, requiring a constant tracking speed of $15^\circ/\text{hr}$ with no sudden azimuthal acceleration.</i>	1 1 1 1

T08. Evolved Star Spirals (25 points)



The red giant AFGL 3068 is located about 1.0 kpc from Earth. Its images reveal a nearly Archimedean spiral with a constant arm spacing, described by $r = a + b\phi$ where a and b are constants, and ϕ is azimuthal angle in radian.

Such a spiral structure can be formed around a binary system. A red giant with a fast stellar wind orbiting the binary center of mass produces a thin, enhanced-density shell that expands radially at the wind velocity of V_w in the center of mass coordinate system. Its shapes on both the orbital plane (Oxy) and a perpendicular plane (Oxz) are shown below, where ϕ and θ denote the azimuthal and polar angles, respectively. Let r_p and V_p be the orbital radius and velocity of the red giant, respectively.



Part 1 (10 points):

a) (2 points) The on-sky arm spacing of AFGL 3068 is 2.29 arcsec. Determine the constant b in au rad^{-1} .

Solution	Point
The spiral arm spacing of AFGL 3068 is $\Delta r_{\text{arm}}[\text{au}] = \Delta \theta[\text{arcsec}] \times d[\text{pc}] = 2.29 \times 1000 = 2290$ When the angle ϕ increases by 2π, the radius r increases by an amount equal to the arm spacing. Therefore, we have $b = \frac{\Delta r_{\text{arm}}}{2\pi} = \frac{2290}{2\pi} \approx 364 \text{ au rad}^{-1}$	1 1

b) (4 points) At every point on the red giant trajectory, we consider only the wind moving radially outward with respect to the center of the binary system. Show that, on the orbital plane, the shell forms an Archimedean spiral structure with $a = r_p$ and $b = \frac{r_p V_w}{V_p}$.

Solution	Point
Consider a shell element located on the orbital plane. Because it moves radially outward with velocity V_w , its radial distance at a time t is	
$r(t) = r_p + V_w t$	1
At the same time, the red giant rotates an angle	
$\phi(t) = \frac{V_p}{r_p} t$	1
Extracting t from this equation and substituting it into the previous equation yields	
$r(t) = r_p + \frac{r_p V_w}{V_p} \phi(t)$	1
On the orbital plane, shell elements all obey an equation of the form given above, but with different values of ϕ , as they are ejected at different times. At the observation time, these elements trace an Archimedean spiral described by	
$r = r_p + \frac{r_p V_w}{V_p} \phi$	1
$\Rightarrow a = r_p, \quad b = \frac{r_p V_w}{V_p}$	

c) (2 points) The wind velocity of AFGL 3068 is $V_w = 14.0 \text{ km s}^{-1}$. Calculate its orbital period T in years.

Solution	Point
From b), the constant b is	

$b = \frac{r_p V_w}{V_p}$ The orbital period of the red giant is given by $T = \frac{2\pi r_p}{V_p} = \frac{2\pi b}{V_w}$ Substituting $V_w = 14.0 \text{ km s}^{-1} \approx 2.95 \text{ au yr}^{-1}$ and $b = 364 \text{ au rad}^{-1}$ into the above formula yields $T = \frac{2\pi \times 364}{2.95} \approx 775 \text{ yr}$	1 1
---	-------------------

d) (2 points) The total mass of the binary system is $6.6 M_{\odot}$. Calculate the separation d between the two stars in au.

Solution	Point
Applying Kepler's third law to the binary system, the separation between the two stars is given by	
$d^3 = M_{\text{tot}} T^2$	1
$\Rightarrow d = \sqrt[3]{M_{\text{tot}} T^2} = \sqrt[3]{6.6 \times 775^2} \approx 158 \text{ au}$	1

Part 2 (15 points): Assume that the orbital plane is inclined by an angle i with respect to the sky plane. The x -axis is along the intersection of the orbital and sky planes. r_p is negligible compared with the spiral size.

a) (6 points) Express the coordinates (x, y, z) and velocities (V_x, V_y, V_z) of a shell element in terms of the parameters (ϕ, θ) .

Solution	Point
The coordinates of a point on the shell can be written as	
$x = r \sin \theta \cos \phi = \frac{r_p V_w}{V_p} \phi \sin \theta \cos \phi$	1.5

$y = r \sin \theta \sin \phi = \frac{r_p V_w}{V_p} \phi \sin \theta \sin \phi$	1.5
$z = r \cos \theta = \frac{r_p V_w}{V_p} \phi \cos \theta$	1.5
Because all shell elements move radially outward with velocity V_w , the components of the velocity in the Cartesian coordinates are given by	
$V_x = V_w \sin \theta \cos \phi$	0.5
$V_y = V_w \sin \theta \sin \phi$	0.5
$V_z = V_w \cos \theta$	0.5

b) (7 points) Show that all shell elements with line-of-sight velocity $V_{\text{los}} = 0$ satisfy the equation of an ellipse on the sky plane with two semi-axes of $r \sin \theta$ and $r \sin \theta / \cos i$.

Solution	Point
Choose the sky plane coordinates (X, Y) with the X -axis identical to the x -axis. The line-of-sight velocity is	
$V_{\text{los}} = V_y \sin i - V_z \cos i = V_w (\sin \theta \sin \phi \sin i - \cos \theta \cos i)$	2
From $V_{\text{los}} = 0$, we have	
$\sin \theta \sin \phi \sin i = \cos \theta \cos i$	1
The sky plane coordinates (X, Y) of a shell element are given by	
$X = x = r \sin \theta \cos \phi$	1
$Y = y \cos i + z \sin i = r (\sin \theta \sin \phi \cos i + \cos \theta \sin i)$	1
Substituting $\cos \theta = \sin \theta \sin \phi \tan i$ into Y coordinate,	
$Y = r \sin \theta \sin \phi (\cos i + \tan i \sin i) = \frac{r \sin \theta \sin \phi}{\cos i}$	1
The sky plane coordinates (X, Y) satisfy	

$\frac{X^2}{(r \sin \theta)^2} + \frac{Y^2}{\left(\frac{r \sin \theta}{\cos i}\right)^2} = 1$	1
---	---

c) (2 points) Recent CO line images at $V_{\text{los}} = 0$ channel show that the spiral pattern of AFGL 3068 is slightly elongated relative to an Archimedean spiral. The elongated pattern is well described by ellipses with a ratio of 1.1 between the two semi-axes. Determine the inclination angle i .

Solution	Point
From b), the ratio of the semi-major axis and the semi-minor axis is given by $\frac{1}{\cos i}$	1
For the spiral pattern of AFGL 3068, we have $\frac{1}{\cos i} = 1.1 \Rightarrow i \approx 24.6^\circ$	1

T09. Gravitational Lensing and Delay (20 points)

Part 1 (15 points): Einstein Ring

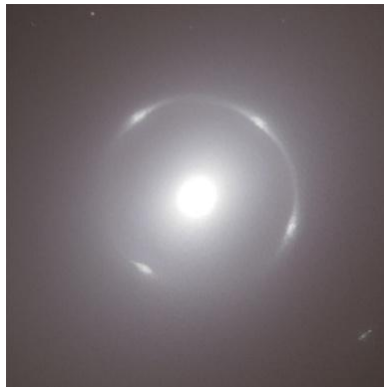


Figure T09.1. Euclid image of NGC 6505 and the Einstein ring surrounding it.

Source: ESA/Euclid/Euclid Consortium/NASA.

In February 2025, the Euclid mission reported an Einstein ring around the nearby elliptical galaxy NGC 6505, located at a distance d_L from Earth (Figure T09.1). The

ring is formed when light from a source galaxy at a distance $d_S \gg d_L$ passes near NGC 6505 and is bent by its gravitational field.

a) (2 points) The measured redshifts of the source and lens galaxies are 0.406 and 0.042, respectively. Calculate d_S and d_L (in Mpc) assuming that the Newtonian approximation is valid and distance is proportional to redshift.

Solution	Point
The redshift z measures the shift in the wavelength of radiation emitted by a source (λ) and observed at a longer wavelength by the observer ($\lambda + \Delta\lambda$); it is caused by the expansion of the Universe between the source and the observer. For nearby objects, we have $z = \frac{\Delta\lambda}{\lambda} \simeq \frac{\Delta V}{c}$ Using Hubble's law gives $\Delta V = H_0 d$ where d is the distance between the observer and the source. Hence, $d = zc/H_0$. This result leads to $d_S = \frac{0.406c}{H_0} = 1800 \text{ Mpc}$ $d_L = \frac{0.042c}{H_0} = 190 \text{ Mpc}$ <i>Note: If students then calculate the angular diameter distance as $d/(1+z)$ they should also receive full points.</i>	0.5 0.5 0.5 0.5

b) (3 points) In the linear approximation as shown in Figure T09.2, the total deflection angle of a light ray emitted by the source and reaching the observer is given by $2\alpha = 2R^*/b$, where R^* is the Schwarzschild radius of the lens and b is the distance of closest approach of the light ray to the center of the lens. Express the angular radius θ of the Einstein ring as a function of R^* , d_S and d_L .

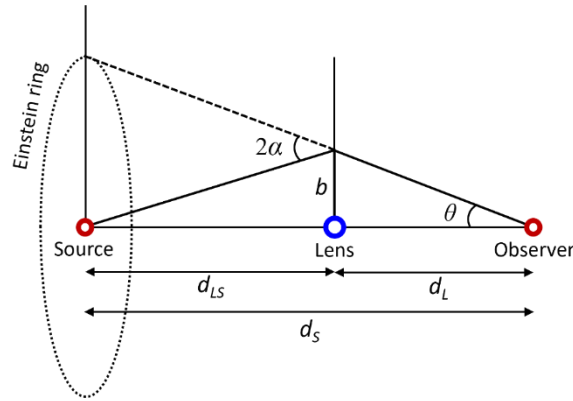


Figure T09.2. Simplified geometry of Einstein ring formation.

Solution	Point
From Figure T09.2, and using the small angle approximation, we obtain	
$d_{LS} = d_S - d_L$	
$\theta d_S = 2\alpha d_{LS}$	1
$\theta d_L = b$	1
Based on the relation $2\alpha = 2R^*/b$, we find	
$2\alpha = \frac{2R^*}{\theta d_L}$	
$\theta d_S = \frac{2R^* d_{LS}}{\theta d_L}$	
$\theta^2 = \frac{2R^* d_{LS}}{d_L d_S}$	
$\theta = \sqrt{\frac{2R^* d_{LS}}{d_L d_S}} = \sqrt{\frac{2R^* (d_S - d_L)}{d_L d_S}}$	1
<i>Note: If students include the minus sign in their answer, deduct 0.5 points.</i>	

c) (3 points) Euclid measures the angular radius of the ring around NGC 6505 to be 2.5 arcsec. Determine the distance of closest approach b (in parsecs) and the mass of the lens galaxy M (in units of $M_\odot$).

Determine the quantities x and x' (indicated in the figure), as well as the angular displacement $\Delta\theta$ of the Einstein ring image.

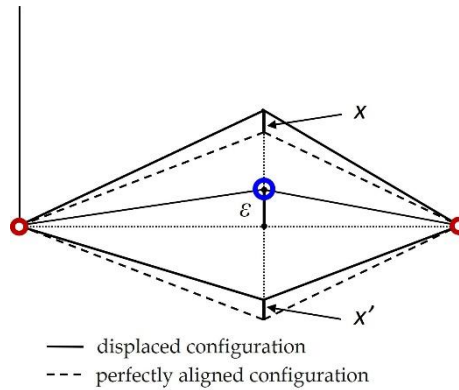


Figure T09.3. Geometry obtained when the lens galaxy is displaced.

Solution	Point
When the lens is displaced by a small distance ε (Figure T09.3), the distance of closest approach decreases for the upper ray, increasing its deflection. The ray that reaches the observer must therefore be emitted at a larger angle. Conversely, for the lower ray, the distance of closest approach increases, reducing the deflection, so the observed ray must be emitted at a smaller angle. Let θ' be the new angular radius of the ring, namely the angle between the detected light ray and the line joining the observer and the lens, $2\alpha'$ be the new deflection angle, and b' be the new distance to the closest approach. First, considering the upper part of the figure, we have $\theta + \frac{x}{d_L} = \theta' + \frac{\varepsilon}{d_L}$ $\Rightarrow x = \theta' d_L - \theta d_L + \varepsilon \quad (1)$ Since $b = \theta d_L$ and $b' = \theta' d_L$, we obtain $x = b' - b + \varepsilon \quad (2)$ Repeating the calculation described in b), we find	1

$2\alpha' d_{LS} = \left(\theta' + \frac{\varepsilon}{d_L} \right) d_S \quad (3)$	0.5
Moreover, $2\alpha' = \frac{2R^*}{\theta' d_L} \quad (4)$	0.5
Hence, $\left(\theta' + \frac{\varepsilon}{d_L} \right) d_S = \frac{2R^*}{\theta' d_L} d_{LS} \quad (5)$	
For the perfectly aligned configuration, we have $\theta d_S = \frac{2R^*}{\theta d_L} d_{LS} \quad (6)$	
Dividing (5) by (6) gives $\frac{\theta'}{\theta} + \frac{\varepsilon}{\theta d_L} = \frac{\theta}{\theta'}$ $\Rightarrow \frac{\theta'}{\theta} + \frac{\varepsilon}{b} = \frac{\theta}{\theta'} \text{ or } \frac{b'}{b} + \frac{\varepsilon}{b} = \frac{b}{b'} \quad (7)$	1
To simplify the calculation, we define $\frac{\theta'}{\theta} = 1 + k \frac{\varepsilon}{b} \quad (8)$	
Since $\varepsilon/b \ll 1$, Eq. (7) reduces to $1 + k \frac{\varepsilon}{b} + \frac{\varepsilon}{b} = 1 - k \frac{\varepsilon}{b} \Rightarrow k = -\frac{1}{2} \quad (9)$	
Substituting k into Eq. (8), we obtain $\frac{\theta'}{\theta} = 1 - \frac{\varepsilon}{2b} \quad (10)$	
Hence, $\frac{b'}{b} = 1 - \frac{\varepsilon}{2b} \Rightarrow b' - b = -\frac{\varepsilon}{2} \quad (11)$	

Note: Alternatively, Eq. (7) can be solved as follows:

Let $t = \theta'/\theta = b'/b$, we have

$$t^2 + \frac{\varepsilon}{b}t - 1 = 0$$

This equation has two solutions, but we retain only the positive one, which is physically meaningful. The positive solution is

$$t = \frac{-\varepsilon/b + \sqrt{(\varepsilon/b)^2 + 4}}{2}$$

Since $\varepsilon/b \ll 1$, we can keep only the first-order term in ε/b ,

$$t = 1 - \frac{\varepsilon}{2b} \Rightarrow b' - b = -\frac{\varepsilon}{2}$$

Substituting this result into Eq. (2), we finally get

$$x = \frac{\varepsilon}{2} \quad (12) \quad 2$$

For the lower part of the figure, we obtain a similar result by changing ε to $-\varepsilon$ in corresponding equations:

$$x' = \frac{\varepsilon}{2} \quad (13) \quad 1$$

The angular displacement of the ring image is

$$\Delta\theta = \frac{\varepsilon}{2d_L} \quad (14) \quad 1$$

To first order, the light rays in the plane perpendicular to the plane of Figure T09.3 are unaffected. Hence, the observer sees the Einstein ring displaced in the same direction as the lens, but by only *half the lens displacement*.

Part 2 (5 points): Gravitational Delay

Let's now consider light being lensed by the Sun.

a) (2 points) Calculate the total deflection angle 2α (in arcsec) of a light ray grazing

the Sun's surface.

Solution	Point
Since the light ray approaching the Sun tangentially, the distance of closest approach is $b = R_{\odot}$ Hence, the total deflection angle is $2\alpha = \frac{2R^*}{b} = \frac{2R^*}{R_{\odot}}$ The Schwarzschild radius of the Sun is given by $R^* = \frac{2GM_{\odot}}{c^2} = 2.95 \text{ km } (\sim 3 \text{ km})$ Substituting the values of R^* and $R_{\odot}$, we obtain $2\alpha = \frac{2 \times 2.95}{6.96 \times 10^5} = 0.85 \times 10^{-5} [\text{rad}] = 1.75 [\text{arcsec}]$ <i>(1.75 – 1.78 arcsec for full mark)</i>	0.5 0.5 0.5 0.5

b) (3 points) A light ray propagating in the gravitational field of a massive object experiences an additional travel time, known as the gravitational delay (δ).

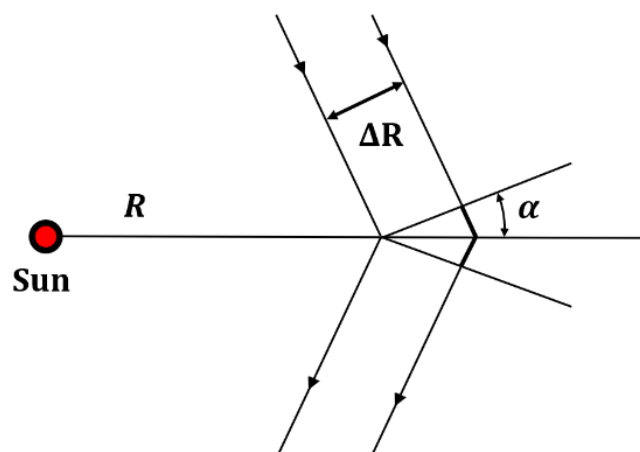


Figure T09.4. Simplified geometry of light bending and gravitational delay.

Consider two light rays passing near the Sun with distances of closest approach

(R) differing by a small amount ΔR (Figure T09.4). The ray with the larger distance of closest approach travels an additional path length of approximately $2\alpha\Delta R$, resulting in a difference in gravitational delay $\Delta\delta \sim 2\alpha\Delta R/c$. Therefore, the absolute value of the derivative of the gravitational delay is

$$\frac{d\delta}{dR} = \frac{2\alpha}{c}$$

Determine the gravitational delay for a light signal from a distant source passing near the Sun's surface and reaching the Earth.

Solution	Point
The relation $d\delta/dR = 2\alpha/c$ was derived using the small-angle geometry for light rays passing close to the Sun. This approximation becomes less accurate as the distance from the Sun increases. Nevertheless, for this simplified exercise, we use it to estimate the total gravitational delay. The gravitational delay experienced by a light ray between the Sun surface ($R = R_{\odot}$) and the Earth ($R = d_{\odot} = 1 \text{ au}$) is the integral of the derivative between these limits, namely $\delta = \int_{R_{\odot}}^{d_{\odot}} \frac{2\alpha}{c} dR$ The total bending angle of a light ray grazing the Sun is $2\alpha = \frac{2R^*}{R}$ Hence, the gravitational delay is $\delta = \int_{R_{\odot}}^{d_{\odot}} \frac{2R^*}{cR} dR = \frac{2R^*}{c} \int_{R_{\odot}}^{d_{\odot}} \frac{1}{R} dR$ $\Rightarrow \delta = \frac{2R^*}{c} \ln\left(\frac{d_{\odot}}{R_{\odot}}\right)$	1 0.5 1

$\Rightarrow \delta = \frac{2 \times 2.95}{3 \times 10^5} \ln \left(\frac{1.496 \times 10^8}{6.96 \times 10^5} \right) = 106 \mu\text{s}$ <i>Note: If students integrate from the solar radius $R_\odot$ to any $R' < 1 \text{ au}$, they will also receive full credit.</i>	0.5
---	-----

T10. The Chevallier-Polarski-Linder Parametrization for Dynamical Dark Energy (20 points)

Consider dynamical dark energy, which is parameterized by an equation of state,

$$w_{\text{DE}}(z) = \frac{P_{\text{DE}}}{c^2 \rho_{\text{DE}}}$$

where $w_{\text{DE}}(z)$ is a function of redshift z , while P_{DE} and ρ_{DE} are the pressure and density of dark energy, respectively.

a) (5 points) In the spatially flat Friedmann-Lemaitre-Robertson-Walker spacetime, the equation of conservation of dark energy is

$$\dot{\rho}_{\text{DE}} + 3H \left(\rho_{\text{DE}} + \frac{P_{\text{DE}}}{c^2} \right) = 0$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter, $a(t)$ is the scale factor, and t is the cosmic time. Hereafter, an overdot denotes the time derivative of a quantity. Show that

$$\rho_{\text{DE}}(z) = \rho_{\text{DE},0} \exp \left[\int_0^z \frac{3[1 + w_{\text{DE}}(z')]}{1 + z'} dz' \right]$$

where $\rho_{\text{DE},0} = \rho_{\text{DE}}(t_0)$, provided that $a(t_0)/a(t) = 1 + z$, with t_0 being the present cosmic time. And $a(t_0) = 1$ will be chosen for convenience.

Solution	Point
Rewrite the equation of conservation of dark energy as follows $\frac{d\rho_{\text{DE}}}{dt} + 3 \left(\frac{1}{a} \frac{da}{dt} \right) (1 + w_{\text{DE}}) \rho_{\text{DE}} = 0$ $\Rightarrow \frac{d\rho_{\text{DE}}}{\rho_{\text{DE}}} = -3(1 + w_{\text{DE}}) \frac{da}{a}$	1

Based on the relation,

$$\frac{1}{a} = 1 + z \Rightarrow \frac{da}{a} = -adz = -\frac{dz}{z+1}$$

we obtain

$$\frac{d\rho_{DE}}{\rho_{DE}} = 3[1 + w_{DE}(z)] \frac{1}{1+z} dz$$

Integrate both sides,

$$\int_{\rho_{DE,0}}^{\rho_{DE}} \frac{d\rho'_{DE}}{\rho'_{DE}} = \int_0^z \frac{3[1 + w_{DE}(z')]}{1+z'} dz'$$

where the prime indicates the integration variables.

Compute the left-hand integral,

$$\ln\left(\frac{\rho_{DE}}{\rho_{DE,0}}\right) = \int_0^z \frac{3[1 + w_{DE}(z')]}{1+z'} dz'$$

Finally, we have

$$\rho_{DE}(z) = \rho_{DE,0} \exp\left[\int_0^z \frac{3[1 + w_{DE}(z')]}{1+z'} dz'\right]$$

b) (5 points) Consider the Friedmann equation,

$$3H^2 = 8\pi G(\rho_m + \rho_{DE})$$

where ρ_m is the density of matter (ordinary and dark matter). When the pressure of matter is negligible, the equation of conservation of matter is

$$\dot{\rho}_m + 3H\rho_m = 0$$

Show that the evolution parameter $E(z) = H(z)/H_0$ obeys

$$E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} \exp\left\{\int_0^z \frac{3[1 + w_{DE}(z')]}{1+z'} dz'\right\}$$

where $\Omega_{m,0} = \rho_{m,0}/\rho_{crit,0}$, $\Omega_{DE,0} = \rho_{DE,0}/\rho_{crit,0}$, $\rho_{crit,0} = 3H_0^2/(8\pi G)$, $H_0 = H(t_0)$, and $\rho_{m,0} = \rho_m(t_0)$.

Solution	Point
From the equation of conversation of matter, we get $\dot{\rho}_m + 3H\rho_m = 0$ $\Rightarrow \frac{d\rho_m}{dt} + \frac{3}{a} \frac{da}{dt} \rho_m = 0$ $\Rightarrow \frac{d\rho_m}{\rho_m} = \frac{3}{1+z} dz$ Integrate both sides, $\int_{\rho_{m,0}}^{\rho_m} \frac{d\rho'_m}{\rho'_m} = \int_0^z \frac{3}{1+z'} dz'$ We obtain $\rho_m = \rho_{m,0}(1+z)^3$ Insert ρ_m and ρ_{DE} into the Friedmann equation and divide both sides of this equation by $3H_0^2$: $\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2} \left\{ \rho_{m,0}(1+z)^3 + \rho_{DE,0} \exp \int_0^z \frac{3[1+w_{DE}(z')]}{1+z'} dz' \right\}$ Finally, we get $E^2(z) = \frac{\rho_{m,0}}{\rho_{crit,0}} (1+z)^3 + \frac{\rho_{DE,0}}{\rho_{crit,0}} \exp \left\{ \int_0^z \frac{3[1+w_{DE}(z')]}{1+z'} dz' \right\}$ $\Rightarrow E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} \exp \left\{ \int_0^z \frac{3[1+w_{DE}(z')]}{1+z'} dz' \right\}$ <i>Note: Students can directly write down the formula of ρ_m without derivations. The marks remain unchanged.</i>	1 1 1 1 1

c) (5 points) Consider the Chevallier-Polarski-Linder parametrization,

$$w_{DE}(a) = w_0 + w_a(1-a)$$

where w_0 and w_a are coefficients. Notice that $a(t_0) = 1$. Find the corresponding $E^2(z)$.

Solution	Point
Rewrite $E(z)$ as $E(a)$, $E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} \exp \left\{ \int_0^z \frac{3[1 + w_{DE}(z')]}{1+z'} dz' \right\}$ $\Rightarrow E^2(a) = \Omega_{m,0}a^{-3} + \Omega_{DE,0} \exp \left\{ - \int_1^a \frac{3[1 + w_{DE}(a')]}{a'} da' \right\}$ Insert $w_{DE}(a) = w_0 + w_a(1-a)$ into the above equation, $E^2(a) = \Omega_{m,0}a^{-3} + \Omega_{DE,0} \exp \left\{ - \int_1^a \frac{3[1 + w_0 + w_a(1-a')]}{a'} da' \right\}$ $= \Omega_{m,0}a^{-3} + \Omega_{DE,0} \exp \left\{ \int_1^a 3w_a da' - \int_1^a \frac{3[1 + (w_0 + w_a)]}{a'} da' \right\}$ $= \Omega_{m,0}a^{-3} + \Omega_{DE,0} e^{3w_a(a-1)} a^{-3[1+(w_0+w_a)]}$ Rewrite $E(a)$ as $E(z)$ $E^2(z) = \Omega_{m,0}(1+z)^3 + \Omega_{DE,0} e^{-\frac{3w_a z}{1+z}} (1+z)^{3[1+(w_0+w_a)]}$ <i>Note: Students can directly compute $E(z)$ by substituting $a = 1/(1+z)$ into w_{DE}. The marks remain unchanged.</i>	1 3 1

d) (5 points) Matter clumps when it dominates over dark energy. Determine the necessary condition on $w_0 + w_a$ such that matter can clump in early times at high redshifts $z \gg 1$.

Solution	Point
Matter clumps in early times at high redshifts $z \gg 1$ if it dominates over the dynamical dark energy. Mathematically, this condition can be stated that the first term on the right-hand side of the expression of $E^2(z)$ must be much larger than the second one, i.e., $\Omega_{m,0}(1+z)^3 \gg \Omega_{DE,0} e^{-\frac{3w_a z}{1+z}} (1+z)^{3[1+(w_0+w_a)]}$ Point out, when $z \gg 1$,	2

$-\frac{3w_a z}{1+z} \approx -3w_a$ The condition becomes $\frac{\Omega_{m,0}(1+z)^3}{\Omega_{DE,0} e^{-3w_a(1+z)^{3[1+(w_0+w_a)]}}} \gg 1$ $\Rightarrow \frac{(1+z)^3}{(1+z)^{3[1+(w_0+w_a)]}} \gg 1$ $\Rightarrow (1+z)^{3(w_0+w_a)} \ll 1$ $\Rightarrow w_0 + w_a < 0$ <i>Note: DESI and CMB show that $w_0 = -0.45$ and $w_a = -1.79$, which derivate 2.2σ from the standard ΛCDM model predictions. Statistical significance increases to 3.9σ when combined with other measurements (BAO and SN Ia) [JCAP 02 (2025) 021].</i>	1 1 1
--	----------------------------

T11. Physics of Jupiter's Interior (50 points)

Jupiter is the biggest planet in our Solar System. Its mass and radius are $M_J = 1.90 \times 10^{27}$ kg and $R_J = 6.99 \times 10^7$ m, respectively. Herein, we investigate Jupiter's internal structure using a simple model. As shown in Figure T11.1, this gas giant is assumed to contain two distinct regions: a dense core and a fluid envelope. While the dense core is entirely made of rock, the fluid envelope is wholly composed of hydrogen. The mass densities of rock (ρ_Z) and hydrogen (ρ_H) are given by

$$\rho_Z = \text{const}, \quad \rho_H = \sqrt{\frac{P}{K}}$$

where P is the hydrostatic pressure and K is the polytropic coefficient. The space outside Jupiter is considered an absolute vacuum. The effects of self-rotation are neglected to achieve perfect spherical symmetry. These approximations enable us to study Jupiter's structural properties via a single spatial variable: the distance r from the center O .

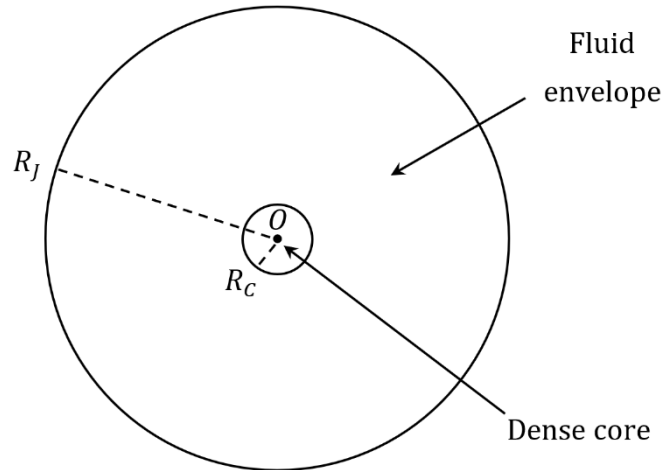


Figure T11.1. Preliminarily modeling Jupiter's interior.

Part 1 (23 points):

Assume that the dense core is not eroded by the fluid envelope. In this case, P decreases from $P_0 = 7.13 \times 10^{12}$ Pa at the center ($r = 0$) to $P_c = 3.98 \times 10^{12}$ Pa at the core-envelope boundary ($r = R_c = 1.05 \times 10^7$ m). The polytropic coefficient of pure hydrogen is $K = 2.12 \times 10^5 \text{ m}^5 \text{ kg}^{-1} \text{ s}^{-2}$.

a) (6 points) Find the value of ρ_Z .

Solution	Point
Let us consider a spherical shell of radius r and thickness dr inside the dense core. Its hydrostatic equilibrium is described by	
$dP = -\rho_Z g dr \quad (1)$	2
where dP is the differential of pressure. The gravitational acceleration g is given by	
$g = \frac{4}{3} \pi G \rho_Z r \quad (2)$	1
where G is the gravitational constant. By inserting Eq. (2) into Eq. (1), we have	
$dP = -\frac{4}{3} \pi G \rho_Z^2 r dr \quad (3)$	1

Integrating both sides of Eq. (3) provides

$$P = -\frac{2}{3}\pi G \rho_Z^2 r^2 + P_O \quad (4) \quad 1$$

When r increases from 0 to R_C , P decreases from P_O to P_C . Therefore,

$$\rho_Z = \frac{1}{R_C} \sqrt{\frac{3(P_O - P_C)}{2\pi G}} = 1.43 \times 10^4 \text{ kg m}^{-3} \quad (5) \quad 1$$

b) (6 points) Since the fluid envelope is equivalent to a simple polytrope ($P \propto \rho_H^2$), its density depends on the distance as

$$\rho_H = \frac{A}{r} \cos\left(\frac{r}{R} + \varphi\right)$$

where R is the characteristic radius, A is the amplitude factor, and φ is the phase constant. Given that

$$R = 2.25 \times 10^7 \text{ m}, \quad A > 0, \quad -\pi \leq \varphi \leq \pi$$

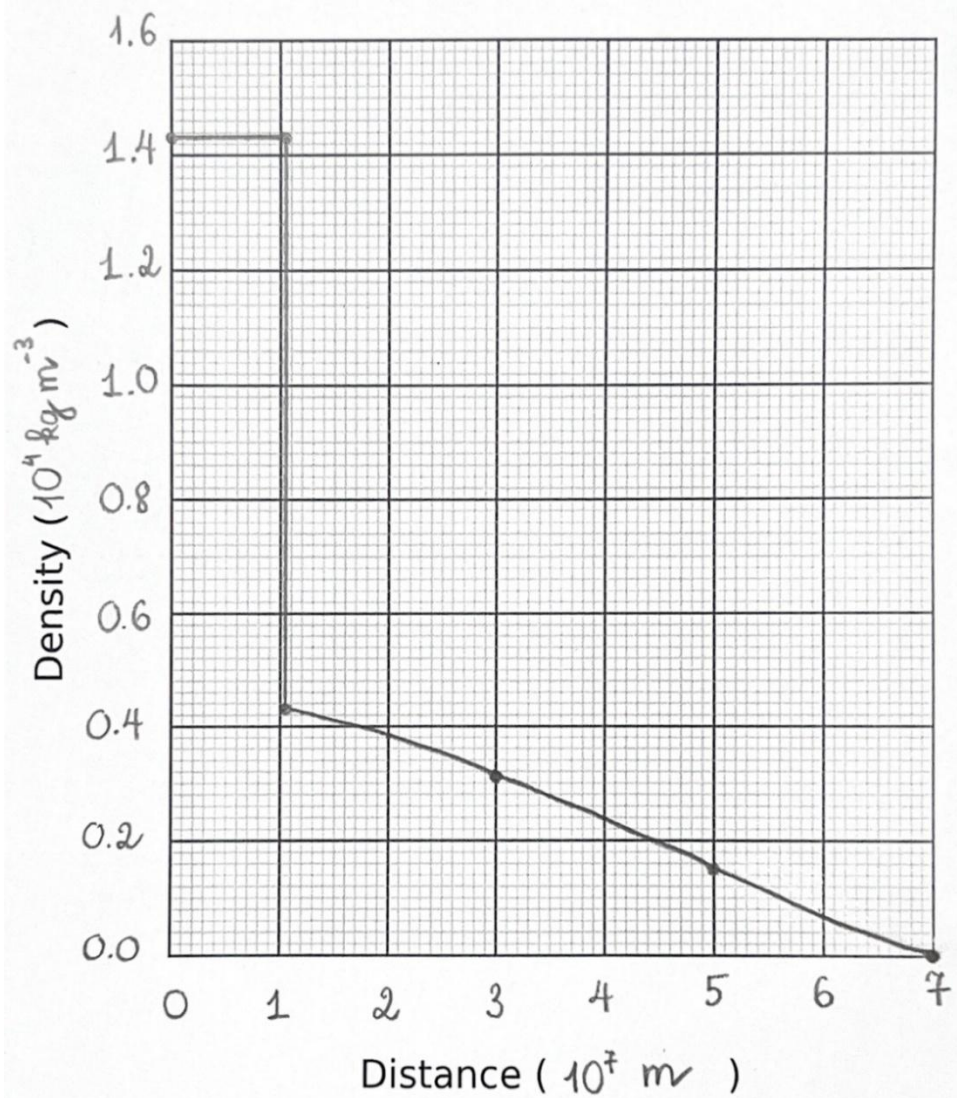
Determine A and φ .

Solution	Point
Because Jupiter is surrounded by a perfect vacuum, the hydrostatic pressure must be equal to zero at the planetary surface. That is synonymous with $\rho_H(R_J) = \frac{A}{R_J} \cos\left(\frac{R_J}{R} + \varphi\right) = 0 \quad (6)$ In addition, at the core-envelope boundary, the mass density of the fluid envelope is $\rho_H(R_C) = \frac{A}{R_C} \cos\left(\frac{R_C}{R} + \varphi\right) = \sqrt{\frac{P_C}{K}} \quad (7)$ Solving Eq. (6) and Eq. (7) with $A > 0$ and $-\pi \leq \varphi \leq \pi$ provides	1 1

$A = \frac{R_C}{\cos\left(\frac{R_C - R_J}{R} + \frac{\pi}{2}\right)} \sqrt{\frac{P_C}{K}} \quad (8)$	1
$\varphi = \frac{\pi}{2} - \frac{R_J}{R} \quad (9)$	1
Numerically, we have	
$A = 9.46 \times 10^{10} \text{ kg m}^{-2} \quad (10)$	1
$\varphi = -1.54 \text{ rad} \quad (11)$	1
<i>Note: If the numerical value of the amplitude factor A is fall in a range of 9.46×10^{10} to $9.53 \times 10^{10} \text{ kg m}^{-2}$, the marks remain unchanged.</i>	

c) (8 points) Plot the density and pressure versus the distance for the whole of Jupiter's interior. Each graph must consist of at least six distinct points.

Solution	Point
The density profile	
- For the dense core: - Plot two critical points at $(0, \rho_Z)$ and (R_C, ρ_Z). - Plot a straight line passing through these points. - For the fluid envelope: - Plot two critical points at $(R_C, \rho_H(R_C))$ and $(R_J, 0)$. - Plot at least two additional points between $r = R_C$ and $r = R_J$ using $\rho_H = \frac{A}{r} \cos\left(\frac{r}{R} + \varphi\right)$. - Plot an appropriate smooth curve passing through these points.	1 0.5 1 1 0.5
<i>Note: It is optional to plot a straight line passing through two critical points (R_C, ρ_Z) and $(R_C, \rho_H(R_C))$.</i>	



The pressure profile

- For the dense core:

Plot two critical points at $(0, P_0)$ and (R_C, P_C) .

1

- For the fluid envelope:

- Plot one critical point at $(R_J, 0)$.

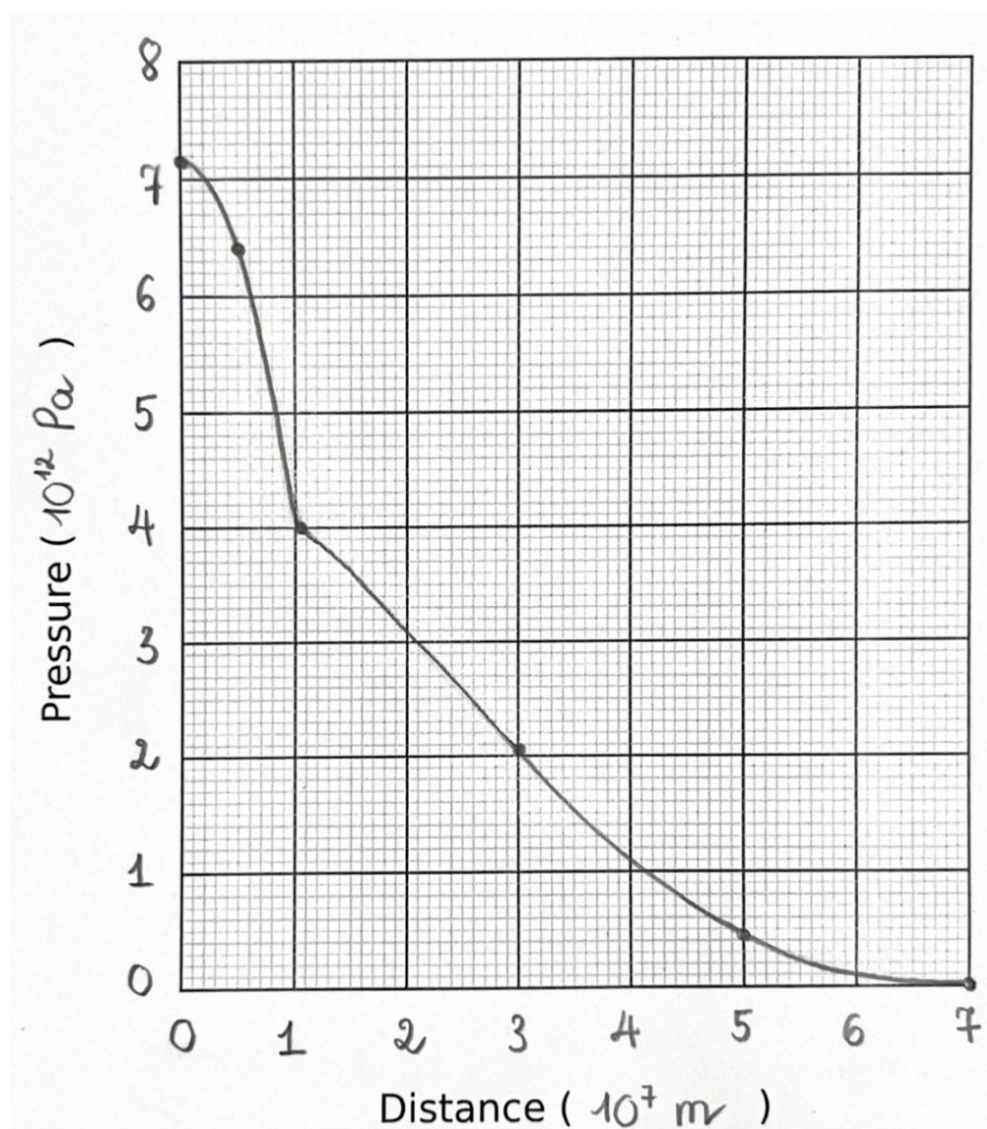
0.5

- Plot at least two additional points between $r = R_C$ and $r = R_J$ using $P = K\rho_H^2$.

1

- Graph finalization:

- | | |
|--|-----|
| - Plot one more point in a range $0 \leq r \leq R_c$ [using Eq. (4)] or $R_c \leq r \leq R_j$ [using $P = K\rho_H^2$] | 0.5 |
| - Plot an appropriate smooth curve passing through all the points representing the dense core (parabola with vertex on y -axis). | 0.5 |
| - Plot an appropriate smooth curve passing through all the points representing the fluid envelope. | 0.5 |



Note: Deduct 0.5 points for each graph missing axis units.

d) (3 points) Calculate the mass of the dense core M_z and the mass of the fluid envelope M_H .

Solution	Point
The mass of the dense core is $M_Z = \frac{4}{3}\pi R_c^3 \rho_Z = 6.93 \times 10^{25} \text{ kg} \quad (12)$	2
The mass of the fluid envelope is $M_H = M_J - M_Z = 1.83 \times 10^{27} \text{ kg} \quad (13)$	1

Part 2 (27 points):

Recent studies show that giant planets may undergo core erosion due to the high solubility of rock in hydrogen. For Jupiter, this process helps redistribute heavy and light materials throughout its interior. When the dense core disappears, Jupiter becomes an ideal mixture of rock and hydrogen at equilibrium. Neglect all possible chemical reactions. For a given position inside Jupiter, the mass fraction of rock Z is defined by

$$Z = \frac{dm_Z}{dm_Z + dm_H}$$

where dm_Z and dm_H are the differential masses of rock and hydrogen, respectively.

a) (6 points) Let ρ be the mass density of the mixture at the distance r . Prove that ρ is related to P via

$$\rho = \frac{\rho_Z \sqrt{P}}{Z \sqrt{P} + (1 - Z) \rho_Z \sqrt{K}}$$

Solution	Point
To formulate the equation of state for the ideal binary mixture, we consider an infinitesimal space of volume dV and mass dM. If we set $dM = dm_Z + dm_H$, dV will be decomposed into $dV = dv_Z + dv_H \quad (14)$ where dv_Z and dv_H represent the volumetric contributions of rock and hydrogen, respectively. Based on the physical definition of the mass density, we can write	2

$dV = \frac{dM}{\rho}, \quad dv_Z = \frac{dm_Z}{\rho_Z}, \quad dv_H = \frac{dm_H}{\rho_H} \quad (15)$	1
Note that $dm_Z = ZdM, \quad dm_H = (1 - Z)dM \quad (16)$	1
Thus, we obtain $\frac{1}{\rho} = \frac{Z}{\rho_Z} + \frac{1 - Z}{\rho_H} \quad (17)$	1
Since $\rho_H = \sqrt{P/K}$, Eq. (17) is equivalent to $\rho = \frac{\rho_Z \sqrt{P}}{Z\sqrt{P} + (1 - Z)\rho_Z \sqrt{K}} \quad (18)$	1

b) (7 points) Figure T11.2 shows how Z depends on r . Based on the $Z - r$ relation, we can divide Jupiter's interior into three different regions: the dilute core ($Z = Z_1 = 7.28 \times 10^{-2}$, $0 \leq r \leq R_1$), the transition layer (Z decreases linearly with increasing r , $R_1 \leq r \leq R_2$), and the hydrogen-dominated envelope ($Z = Z_2 = 1.53 \times 10^{-2}$, $R_2 \leq r \leq R_J$). In the transition layer, ρ is well approximated as a linear function of r , and P drops from $P_1 = 2.25 \times 10^{12}$ Pa at $R_1 = 2.87 \times 10^7$ m to $P_2 = 8.43 \times 10^{11}$ Pa at $R_2 = 4.40 \times 10^7$ m. The polytropic coefficient of hydrogen in the mixture is $K = 2.20 \times 10^5$ m⁵ kg⁻¹ s⁻². Determine the mass of the transition layer M_{12} .

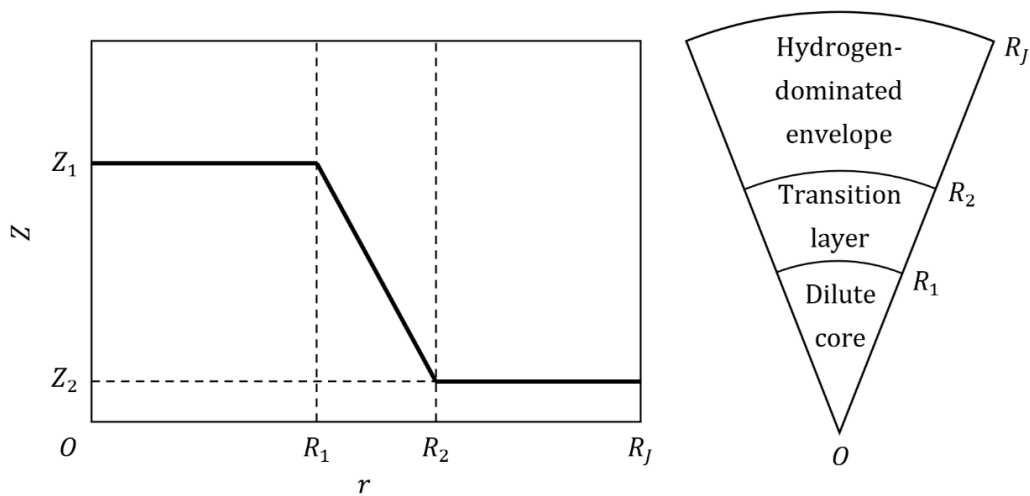


Figure T11.2. (Left) The mass fraction of rock versus the distance from the center.
(Right) The cross-sectional view of Jupiter after core erosion.

Solution	Point
Since ρ is a linear function of r at $R_1 \leq r \leq R_2$, we can write $\rho = \frac{\rho_1 - \rho_2}{R_1 - R_2} r + \frac{\rho_2 R_1 - \rho_1 R_2}{R_1 - R_2} \quad (19)$ Here, ρ_1 and ρ_2 are the critical densities at the base and the top of the transition layer. These quantities are deduced from Eq. (18) as $\rho_1 = \frac{\rho_Z \sqrt{P_1}}{Z_1 \sqrt{P_1} + (1 - Z_1) \rho_Z \sqrt{K}} = 3.39 \times 10^3 \text{ kg m}^{-3}$ $\rho_2 = \frac{\rho_Z \sqrt{P_2}}{Z_2 \sqrt{P_2} + (1 - Z_2) \rho_Z \sqrt{K}} = 1.98 \times 10^3 \text{ kg m}^{-3} \quad (20)$ Having established the $\rho - r$ relation, it is feasible to calculate the mass of the transition layer by $M_{12} = \int_{R_1}^{R_2} 4\pi r^2 \rho dr \quad (21)$ Entering Eq. (19) into Eq. (21) yields $M_{12} = \frac{\pi}{3} (R_2 - R_1) [R_1^2 (3\rho_1 + \rho_2) + R_2^2 (\rho_1 + 3\rho_2) + 2R_1 R_2 (\rho_1 + \rho_2)] \quad (22)$ Our numerical result based on Eq. (20) and Eq. (22) is $M_{12} = 6.67 \times 10^{26} \text{ kg} \quad (23)$	1 1 1 1 2 1

c) (14 points) Calculate the mass of the dilute core M_1 and the mass of the hydrogen-dominated envelope M_2 .

Solution	Point
It is clear that Jupiter's mass is a conservation quantity, $M_1 + M_{12} + M_2 = M_J \quad (24)$	1

When the dense core is eroded, heavy and light materials are mixed without chemical reactions. Hence, the total mass of rock must also be conserved,

$$Z_1 M_1 + M_{12}^{\text{rock}} + Z_2 M_2 = M_Z \quad (25) \quad 3$$

Solving Eq. (24) and Eq. (25) yields

$$M_1 = \frac{M_Z - M_{12}^{\text{rock}} - Z_2(M_J - M_{12})}{Z_1 - Z_2} \quad 1$$

$$M_2 = \frac{Z_1(M_J - M_{12}) - M_Z + M_{12}^{\text{rock}}}{Z_1 - Z_2} \quad (26) \quad 1$$

To determine M_1 and M_2 , we need to find the amount of rock in the transition layer M_{12}^{rock} . This quantity is determined by

$$M_{12}^{\text{rock}} = \int_{R_1}^{R_2} 4\pi Z r^2 \rho dr \quad (27) \quad 1$$

In the transition layer, we have

$$Z = \frac{Z_1 - Z_2}{R_1 - R_2} r + \frac{Z_2 R_1 - Z_1 R_2}{R_1 - R_2}, \quad R_1 \leq r \leq R_2 \quad (28) \quad 1$$

Eqs. (19), (27), and (28) enables us to explicitly express M_{12}^{rock} by

$$M_{12}^{\text{rock}} = \frac{\pi}{15} (R_2 - R_1) (\alpha R_1^2 + \beta R_2^2 + 2\gamma R_1 R_2) \quad (29) \quad 3$$

where

$$\begin{aligned} \alpha &= 3Z_1(4\rho_1 + \rho_2) + Z_2(3\rho_1 + 2\rho_2) \\ \beta &= Z_1(2\rho_1 + 3\rho_2) + 3Z_2(\rho_1 + 4\rho_2) \\ \gamma &= 3Z_1\rho_1 + 2Z_2\rho_1 + 2Z_1\rho_2 + 3Z_2\rho_2 \end{aligned} \quad (30)$$

Our numerical result is

$$M_{12}^{\text{rock}} = 2.84 \times 10^{25} \text{ kg} \quad (31) \quad 1$$

Substituting this number into Eq. (26) provides

$M_1 = 3.83 \times 10^{26} \text{ kg}$	1
$M_2 = 8.50 \times 10^{26} \text{ kg} \quad (32)$	1

T12. Superluminous Galaxies in the Early Universe (50 points)

JWST has identified exceptionally compact, luminous galaxies at redshifts $z > 7$. To understand the nature of these sources, we must analyze the observational data and test whether they are consistent with basic astrophysical constraints. The table below lists the constants and parameters used in this problem.

Star formation rate (SFR) to Luminosity conversion factor	$k = 1.0 \times 10^{10} L_{\odot} M_{\odot}^{-1} \text{ yr}$
Effective opacity of free electrons	$\kappa_e = 0.04 \text{ m}^2 \text{ kg}^{-1}$
Effective opacity of the dusty interstellar medium	$\kappa_{\text{dust}} = 1.5 \text{ m}^2 \text{ kg}^{-1}$
Optical-to-bolometric flux correction factor	$BC = F_{\text{bol}}/F_{\text{opt}} = 9.26$

Part 1 (13 points): Cosmological Surface Brightness Dimming

Let L be the intrinsic bolometric luminosity and R be the intrinsic physical radius of a galaxy. The intrinsic surface luminosity density is defined as

$$\Sigma_{\text{int}} = \frac{L}{\pi R^2}$$

A telescope observes this galaxy subtending with a solid angle Ω_{obs} and measures a flux F_{obs} . The observed surface brightness is

$$\Sigma_{\text{obs}} = \frac{F_{\text{obs}}}{\Omega_{\text{obs}}}$$

In cosmology, the luminosity distance (D_L) is defined via

$$F_{\text{obs}} = \frac{L}{4\pi D_L^2}$$

The angular diameter distance (D_A) is defined so that the observed angular radius is

$$\theta = \frac{R}{D_A}$$

a) (10 points) Derive the relation between D_L , D_A and redshift z .

Solution	Point
Consider a photon emitted with energy E_{em}. Due to cosmic expansion, the energy received by the observer is redshifted, $E_{\text{obs}} = \frac{E_{\text{em}}}{1 + z}$	2
Due to cosmological time dilation, an emission time interval dt_{em} is observed as dt_{obs} at the receiver, $dt_{\text{obs}} = (1 + z)dt_{\text{em}}$	2
Combining these two effects, the observed luminosity is attenuated as $L_{\text{obs}} = \frac{L}{(1 + z)^2}$	
By expressing the observed flux F_{obs} via the comoving distance D_M $F_{\text{obs}} = \frac{L_{\text{obs}}}{4\pi D_M^2} = \frac{L}{4\pi D_M^2(1 + z)^2}$	1
and then comparing this to the definition $F_{\text{obs}} = \frac{L}{4\pi D_L^2}$	
we have $D_L = D_M(1 + z)$	1
The angular size of the object is $\theta = \frac{R(1 + z)}{D_M}$	2
By definition, $\theta = \frac{R}{D_A}$	

Therefore,	
$D_A = \frac{D_M}{1+z}$	1
Substituting $D_M = D_A(1+z)$ into the D_L expression yields	
$D_L = D_A(1+z)^2$	1

b) (3 points) Derive an expression for Σ_{obs} as a function of Σ_{int} and redshift z .

Solution	Point
To establish the equation for Σ_{obs} , we start from the definition	
$\Sigma_{\text{obs}} = \frac{F_{\text{obs}}}{\Omega_{\text{obs}}} = \frac{\frac{L}{4\pi D_L^2}}{\frac{\pi R^2}{D_A^2}}$	1
By re-arranging the terms, we obtain	
$\Sigma_{\text{obs}} = \frac{L}{\pi R^2} \cdot \frac{D_A^2}{4\pi D_L^2} = \Sigma_{\text{int}} \cdot \frac{D_A^2}{4\pi D_L^2}$	1
Substituting $D_A^2/D_L^2 = 1/(1+z)^4$ into the equation gives us	
$\Sigma_{\text{obs}} = \frac{\Sigma_{\text{int}}}{4\pi(1+z)^4}$	1

Part 2 (8 points): Dissecting JWST Data

JWST observed an ultraluminous galaxy at a redshift of $z = 7.26$.

- The observed rest-frame optical flux at Earth (attenuated by cosmic dust) is $F_{\text{obs,opt}} = 5.72 \times 10^{-19} \text{ W m}^{-2}$.
- The dust extinction derived from the spectrum is $A_V = 2.79 \text{ mag}$.
- The measured angular radius of the source on the image is $\theta = 0.05 \text{ arcsec}$.

Calculate the intrinsic bolometric surface luminosity density of this galaxy, Σ_{int} , in units of $L_{\odot} \text{ kpc}^{-2}$.

Solution	Point
The intrinsic rest-frame optical flux (dust-corrected) is given by $F_{\text{opt,int}} = F_{\text{opt,obs}} \cdot 10^{\frac{A_V}{2.5}}$	1
Substituting $F_{\text{opt,obs}}$ and A_V values into the equation provides $F_{\text{opt,int}} = 5.72 \times 10^{-19} \times 10^{\frac{2.79}{2.5}} = 7.47 \times 10^{-18} \text{ W m}^{-2}$	1
The intrinsic bolometric flux: $F_{\text{bol}} = F_{\text{opt,int}} \cdot BC = 7.47 \times 10^{-18} \times 9.26 = 6.92 \times 10^{-17} \text{ W m}^{-2}$	1
In addition, after converting θ to radians $\theta_{\text{rad}} = \frac{0.05}{3600} \times \frac{\pi}{180} = 2.424 \times 10^{-7} \text{ rad}$	1
we can calculate the observed solid angle by $\Omega_{\text{obs}} = \pi \theta_{\text{rad}}^2 = 1.846 \times 10^{-13} \text{ sr}$	1
Therefore, $\Sigma_{\text{obs,corr}} = \frac{F_{\text{bol}}}{\Omega_{\text{obs}}} = \frac{6.92 \times 10^{-17}}{1.846 \times 10^{-13}} = 3.75 \times 10^{-4} \text{ W m}^{-2} \text{ sr}^{-1}$	1
Applying the result from Part 1 gives us Σ_{int} in standard SI units as $\Sigma_{\text{int,SI}} = 4\pi(1+z)^4 \Sigma_{\text{obs,corr}}$	
Since $z = 7.26$, we have $(1+z)^4 \approx 4655.0$. Substituting this value into the equation yields $\Sigma_{\text{int,SI}} = 4 \times \pi \times 4655.0 \times 3.75 \times 10^{-4} = 21.94 \text{ W m}^{-2}$	1
We use the following conversion factor $1 \text{ kpc}^2 = (3.086 \times 10^{19} \text{ m})^2 = 9.523 \times 10^{38} \text{ m}^2$ Hence,	

$\Sigma_{\text{int}} = \frac{21.94 \times 9.523 \times 10^{38}}{3.828 \times 10^{26}} = 5.46 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$			1
Half credit lower limit	Full credit range	Half credit upper limit	
$5.35 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.40 \times 10^{13} \text{ to } 5.50 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	$5.55 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$	

Part 3 (18 points): Eddington Dynamical Model for the Star-Forming Disk

Assume the entire intrinsic radiative energy of this galaxy originates from extreme star formation in an optically thick disk of cold gas and dust. Radiation from young stars generates tremendous pressure that drives cosmic dust particles outward, perpendicular to the disk midplane. Due to the extreme density, gas and dust are perfectly dynamically coupled via collisions. The radiation pressure acting on the dust effectively drags the entire gas mass along as a single unified fluid. Simultaneously, gravity pulls this fluid back toward the midplane. Assume the gravitational field of the disk is governed by the total dynamical surface mass density, given as $\Sigma_{\text{tot}} = 2.3 \times 10^{10} \text{ M}_{\odot} \text{ kpc}^{-2}$.

a) (6 points) $\Sigma_{\text{int_max}}$ denotes the disk's Eddington surface luminosity density. Prove the acceleration driven by radiation pressure is $g_{\text{rad}} = \frac{\kappa_{\text{dust}} \Sigma_{\text{int_max}}}{c}$ and the acceleration governed by the disk's gravity is $g_z = 2\pi G \Sigma_{\text{tot}}$.

Solution	Point
The Eddington surface luminosity density is $\Sigma_{\text{int_max}} = \frac{L_{\text{Edd}}}{\pi R^2}$	
The total radiation force exerted on a dust particle is $f = \sigma \frac{\Sigma_{\text{int_max}}}{c}$	
where σ is the dust cross-section given by $\sigma = m_d \kappa_{\text{dust}}$	1

where m_d is the mass of dust particle.

The acceleration driven by radiation pressure is

$$g_{\text{rad}} = \frac{f}{m_d} = \frac{\kappa_{\text{dust}} \Sigma_{\text{int_max}}}{c}$$

1

Based on the Gauss's gravity theorem, we have

$$2\pi R^2 g_z = 4\pi G M_{\text{tot}}$$

1

where M_{tot} is the total disk's mass.

Hence,

$$g_z = 2\pi G \Sigma_{\text{tot}}$$

1

b) (7 points) Calculate the SFR surface density at Eddington limit (Σ_{Edd}) of this disk (in units of $M_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$) such that it remains gravitationally bound against a radiation-driven blowout.

Solution	Point
Based on the balance between radiation pressure and gravity, we can establish the following critical equilibrium condition	
$g_{\text{rad}} = g_z$	1
$\Rightarrow \frac{\kappa_{\text{dust}} \Sigma_{\text{int_max}}}{c} = 2\pi G \Sigma_{\text{tot}}$	
$\Rightarrow \Sigma_{\text{int_max}} = \frac{2\pi G c \Sigma_{\text{tot}}}{\kappa_{\text{dust}}}$	1
The Eddington SFR surface density is derived from the Kennicutt scaling law:	
$\Sigma_{\text{Edd}} = \frac{\Sigma_{\text{int_max}}}{k}$	1
$\Rightarrow \Sigma_{\text{Edd}} = \frac{2\pi G c \Sigma_{\text{tot}}}{k \kappa_{\text{dust}}}$	1

Converting Σ_{tot} to SI units gives us

$$\Sigma_{\text{tot}} = \frac{2.3 \times 10^{10} \times 1.988 \times 10^{30}}{9.523 \times 10^{38}} = 48.01 \text{ kg m}^{-2}$$

0.5

In SI units, $\Sigma_{\text{int_max}}$ is calculated by

$$\Sigma_{\text{int_max}} = \frac{2\pi \times 6.674 \times 10^{-11} \times 2.998 \times 10^8 \times 48.01}{1.5} = 4.02 \text{ W m}^{-2}$$

1

Using unit conversion

$$4.02 \text{ W m}^{-2} = 1.00 \times 10^{13} \text{ L}_{\odot} \text{ kpc}^{-2}$$

0.5

we obtain

$$\Sigma_{\text{Edd}} = \frac{1.00 \times 10^{13}}{10^{10}} = 1000 \text{ M}_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$$

1

Half credit lower limit	Full credit range	Half credit upper limit
980	990 to 1010	1020

c) (5 points) Calculate the ratio $\Sigma_{\text{SFR_actual}}/\Sigma_{\text{Edd}}$, where $\Sigma_{\text{SFR_actual}}$ is the actual SFR surface density of this galaxy. If the observed luminosity is purely stellar in origin, is this system dynamically stable?

Solution	Point
Using Σ_{int} in Part 2 provides	
$\Sigma_{\text{SFR_actual}} = \frac{\Sigma_{\text{int}}}{k}$	1
$\Sigma_{\text{SFR_actual}} = \frac{5.46 \times 10^{13}}{1.0 \times 10^{10}} = 5460 \text{ M}_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$	1
We can compare $\Sigma_{\text{SFR_actual}}$ with Σ_{Edd} ,	
$\frac{\Sigma_{\text{SFR_actual}}}{\Sigma_{\text{Edd}}} = 5.46$	1
<i>It is clear to see that the actual density vastly exceeds the Eddington limit. The disk is highly super-Eddington.</i>	

b) (7 points) Quantitatively evaluate the radiative stability of the system with respect to its observed luminosity. Is this system radiatively stable?

Solution	Point
<i>Spectral analysis reveals a "highly ionized, intensely hot" gas phase. Under this extreme thermal and radiation environment, the interstellar dust grains are completely sublimated (destroyed).</i> <i>Without dust, the radiation pressure now couples solely to free electrons via Thomson scattering. Consequently, the effective opacity κ precipitously drops from $\kappa_{\text{dust}} = 1.5 \text{ m}^2 \text{ kg}^{-1}$ to $\kappa_e = 0.04 \text{ m}^2 \text{ kg}^{-1}$.</i> Assume the same optical thick disk model as above $L_{\text{Edd}} = \frac{2\pi GMc}{\kappa_e}$ $\Sigma_{\text{Edd}} = \frac{2GMc}{\kappa_e R^2} \approx 37500 \text{ M}_{\odot} \text{ yr}^{-1} \text{ kpc}^{-2}$ $\frac{\Sigma_{\text{SFR}_{\text{actual}}}}{\Sigma_{\text{Edd}}} = \frac{5460}{37500} = 0.15$ <i><u>Note:</u> The transition to Thomson opacity (κ_e) increases the medium's threshold for radiation pressure by a factor of $1.5/0.04 = 37.5$. Hence, $\frac{\Sigma_{\text{SFR}_{\text{actual}}}}{\Sigma_{\text{Edd}}} = \frac{5.46}{37.5} = 0.15$. Award full points.</i> Since $\Sigma_{\text{SFR}_{\text{actual}}} < \Sigma_{\text{Edd}}$, the radiation pressure does not exceed the gravitational pull of the SMBH. The system effectively maintains hydrostatic equilibrium. This system is radiatively stable (YES).	2 2 1 2